

ON THE STRUCTURE OF SOME MODULI SPACES OF FINITE FLAT GROUP SCHEMES

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1. INTRODUCTION AND NOTATIONS

Let p be an odd prime and k a finite field of characteristic p . Let $W = W(k)$ be the ring of Witt vectors with coefficients in k and $K_0 = W[\frac{1}{p}]$ its fraction field. We consider a finite, totally ramified extension K/K_0 and denote by $e = [K : K_0]$ the degree of the extension. Let us fix a uniformizer $\pi \in \mathcal{O}_K$ with minimal polynomial $E(u) \in W[u]$ over K_0 . Further we fix an algebraic closure \bar{K} of K .

Let \mathbb{F} be a finite field of characteristic p and $\rho : G_K \rightarrow GL(V_{\mathbb{F}})$ a continuous representation of the absolute Galois group $G_K = \text{Gal}(\bar{K}/K)$ of K in a finite dimensional \mathbb{F} -vector space $V_{\mathbb{F}}$ whose dimension will be denoted by d .

This datum is equivalent to a finite commutative group scheme $\tilde{\mathcal{G}} \rightarrow \text{Spec } K$ with an operation of \mathbb{F} : The \bar{K} -valued points become an \mathbb{F} -vector space with a natural action of G_K and we want $\tilde{\mathcal{G}}(\bar{K})$ and $V_{\mathbb{F}}$ to be isomorphic as $\mathbb{F}[G_K]$ -modules.

If \mathbb{F}' is a finite extension of \mathbb{F} , the representation ρ induces a representation ρ' on $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$.

By the construction in Kisin's article [Ki], there is a projective \mathbb{F} -scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ whose \mathbb{F}' -valued points parametrize the isomorphism classes of finite flat models of $V_{\mathbb{F}'}$, i.e. finite flat group schemes $\mathcal{G} \rightarrow \text{Spec } \mathcal{O}_K$ with an operation of \mathbb{F}' such that the generic fiber of \mathcal{G} is the G_K -representation on $V_{\mathbb{F}'}$ in the above sense.

Our aim is to analyze the structure of (some stratification of) $\mathcal{GR}_{V_{\mathbb{F}},0}$ in the case $d = 2$ and $k = \mathbb{F}_p$.

First we recall some constructions from [Ki], see also [PR2]. We assume $k = \mathbb{F}_p$ to simplify the situation.

For each n let $\pi_n \in \bar{K}$ be a p^n -th root of the uniformizer π such that $\pi_n^p = \pi_{n-1}$ for all n . Define $K_{\infty} = \bigcup_{n \geq 1} K(\pi_n)$ and denote by $G_{K_{\infty}} = \text{Gal}(\bar{K}/K_{\infty})$ the absolute Galois group of K_{∞} .

For each algebraic extension \mathbb{F}' of \mathbb{F} we denote by $\phi : \mathbb{F}'((u)) \rightarrow \mathbb{F}'((u))$ the homomorphism which takes u to its p -th power and which is the identity on the coefficients:

$$\phi\left(\sum_i a_i u^i\right) = \sum_i a_i u^{pi}.$$

Denote by $\text{Mod}_{/\mathbb{F}'((u))}^{\phi}$ the category of finite dimensional $\mathbb{F}'((u))$ -modules M together with a ϕ -linear map $\Phi : M \rightarrow M$ such that the linearization $\text{id} \otimes \Phi : \phi^* M \rightarrow M$ is an isomorphism. The morphisms are $\mathbb{F}'((u))$ -linear maps commuting with Φ . By ([Ki] 1.2.6, Lemma 1.2.7), there is an equivalence of abelian categories

$$\text{Mod}_{/\mathbb{F}'((u))}^{\phi} \longleftrightarrow \left\{ \begin{array}{l} \text{continuous } G_{K_{\infty}}\text{-representations} \\ \text{on finite dimensional } \mathbb{F}'\text{-vector spaces} \end{array} \right\}$$

which preserves the dimensions and is compatible with finite base change \mathbb{F}''/\mathbb{F}' . This is a version with coefficients of the equivalence of categories of Fontaine (cf. [Fo], A3).

Denote by $(M_{\mathbb{F}}, \Phi)$ the d -dimensional $\mathbb{F}((u))$ -vector space with semi-linear endomorphism Φ , associated to the restriction of the Tate-twist $V_{\mathbb{F}}(-1)$ to $G_{K_{\infty}}$ under the above equivalence. By the descriptions in [Ki], the finite flat models $\mathcal{G} \rightarrow \text{Spec } \mathcal{O}_K$ of $V_{\mathbb{F}}$ correspond to $\mathbb{F}[[u]]$ -lattices $\mathfrak{M} \subset M_{\mathbb{F}}$ satisfying $u^e \mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$. Here $\langle \Phi(\mathfrak{M}) \rangle = (\text{id} \otimes \Phi)\phi^* \mathfrak{M}$ is the $\mathbb{F}[[u]]$ -lattice in $M_{\mathbb{F}}$ generated by $\Phi(\mathfrak{M})$.

Under this description the multiplicative group schemes correspond to the lattices \mathfrak{M} such that $\langle \Phi(\mathfrak{M}) \rangle = \mathfrak{M}$ and the étale group schemes correspond to the lattices with $u^e \mathfrak{M} = \langle \Phi(\mathfrak{M}) \rangle$. These lattices will be called multiplicative resp. étale.

This construction is compatible with base change in the following sense. Suppose $\mathfrak{M}_{\mathbb{F}} \subset M_{\mathbb{F}}$ is a $\mathbb{F}[[u]]$ -lattice corresponding to a finite flat model \mathcal{G} of $V_{\mathbb{F}}$. If \mathbb{F}' is a finite extension of \mathbb{F} with $n = [\mathbb{F}' : \mathbb{F}]$, then the $\mathbb{F}'[[u]]$ lattice

$$\mathfrak{M}_{\mathbb{F}'} = \mathfrak{M} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}' \subset M_{\mathbb{F}'} = M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$$

corresponds to the finite flat model $\mathcal{G}' = \mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}'$ of $V_{\mathbb{F}'}$. Here the exterior tensor product $\mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}'$ is the following group scheme: Choose a \mathbb{F} -basis $e_1 \dots e_n$ of \mathbb{F}' . Then $\mathcal{G} \boxtimes_{\mathbb{F}} \mathbb{F}' = \prod_{i=1}^n \mathcal{G}$ and $z \in \mathbb{F}'$ operates via the matrix $A \in GL_n(\mathbb{F})$ describing the multiplication by z on \mathbb{F}' in the fixed \mathbb{F} -basis.

The scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ is constructed as a closed subscheme of the affine Grassmannian $\text{Grass } M_{\mathbb{F}}$ for $GL(M_{\mathbb{F}})$ and its closed points are given by

$$(1.1) \quad \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') = \{\mathbb{F}'[[u]]\text{-lattices } \mathfrak{M} \subset M_{\mathbb{F}'} \mid u^e \mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}\}$$

for every finite extension \mathbb{F}' of \mathbb{F} .

In the following we will forget about the Galois representation and finite flat group schemes and will consider lattices. We will drop the condition $p \neq 2$. All results hold for arbitrary p , except those using the interpretation of the closed points as finite flat group schemes. We will always assume that there exists a finite flat model for $V_{\mathbb{F}}$ at least after extending scalars.

For each \mathbb{Q}_p -algebra embedding $\psi : K \rightarrow \bar{K}_0$ we now fix an integer $v_{\psi} \in \{0, \dots, d\}$. Denote by $\mathbf{v} = (v_{\psi})_{\psi}$ the collection of the v_{ψ} and by $\mathbf{r} = \check{\mathbf{v}}$ the dual partition, i.e. $r_i = \#\{\psi \mid v_{\psi} \geq i\}$.

Kisin constructs closed, reduced subschemes

$$\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \subset \mathcal{GR}_{V_{\mathbb{F}},0}$$

whose \mathbb{F}' -valued points are given by

$$(1.2) \quad \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}(\mathbb{F}') = \{\mathfrak{M} \in \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \mid J(u|_{\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}}) \leq \mathbf{r}\}$$

for a finite extension \mathbb{F}' of \mathbb{F} (cf. [Ki], Prop. 2.4.6). Here $J(u|_{\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}})$ denotes the Jordan type of the nilpotent endomorphism on $\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}$ induced by the multiplication with u . Recall that for $d = 2$

$$(1.3) \quad (a_1, b_1) \leq (a_2, b_2) \Leftrightarrow \begin{cases} a_1 \leq a_2, \\ a_1 + b_1 = a_2 + b_2 \end{cases}$$

for pairs $(a_i, b_i) \in \mathbb{Z}^2$ with $a_i \geq b_i$. The local structure of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is linked to the structure of the local models studied in [PR1]. These schemes are named "closed Kisin varieties" in [PR2].

Kisin conjectures in ([Ki] 2.4.16) that, if $\text{End}_{\mathbb{F}[\text{G}_K]}(V_{\mathbb{F}}) = \mathbb{F}$, the connected components of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ are given by the open and closed subschemes on which both the rank of the maximal multiplicative subobject and the rank of the maximal étale quotient are fixed. In ([Ki], 2.5) he proves this conjecture in the case $d = 2$, $k = \mathbb{F}_p$ and $v_{\psi} = 1$ for all ψ . For $d = 2$ and $v_{\psi} = 1$ for all ψ this result is generalized by Imai to the case of arbitrary k (see [Im]). In this paper we want to analyze the situation in the case $k = \mathbb{F}_p$, $d = 2$ but arbitrary \mathbf{v} . It turns out that the conjecture is not true in general. Our main results are as follows.

For $(a, b) \in \mathbb{Z}^2$ with $a \geq b$, we introduce a locally closed subscheme of the affine Grassmannian

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \subset \text{Grass } M_{\mathbb{F}},$$

with closed points the lattices \mathfrak{M} such that the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} are given by (a, b) .

Theorem 1.1. *Assume that $(M_{\mathbb{F}'}, \Phi) = (M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}', \Phi)$ is simple for all finite extensions \mathbb{F}' of \mathbb{F} .*

(i) *If $\mathcal{G}_{V_{\mathbb{F}}}(a, b) \neq \emptyset$, there exists a finite extension \mathbb{F}' of \mathbb{F} such that*

$$\mathcal{G}_{V_{\mathbb{F}'}}(a, b) = \mathcal{G}_{V_{\mathbb{F}}}(a, b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}_{\mathbb{F}'}^n$$

for $n = \lfloor \frac{a-b}{p+1} \rfloor$.

(ii) *The scheme $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is geometrically connected and irreducible. There exists a finite extension \mathbb{F}' of \mathbb{F} such that $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \mathbb{F}'$ is isomorphic to a Schubert variety in the affine Grassmannian for $GL(M_{\mathbb{F}'})$.*

The dimension of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is either $\lfloor \frac{r_1-r_2}{p+1} \rfloor$ or $\lfloor \frac{r_1-r_2}{p+1} \rfloor - 1$. Here $r_i = \#\{\psi \mid v_{\psi} \geq i\}$.

In the treatment of the reducible case we consider the set $\mathcal{S}(\mathbf{v})$ of isomorphism classes $[M']$ of one dimensional objects in $\text{Mod}_{\mathbb{F}((u))}^{\phi}$ which admit an $\mathbb{F}[[u]]$ -lattice $\mathfrak{M}_{[M']} \subset M'$ such that $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1} \mathfrak{M}_{[M']}$. We will define subschemes

$$X_{[M']}^{\mathbf{v}} \subset \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \mathbb{F}.$$

A lattice defines a closed point of $X_{[M']}^{\mathbf{v}}$ if it admits a Φ -stable subobject isomorphic to $\mathfrak{M}_{[M']}$. A lattice \mathfrak{M} is called *\mathbf{v} -ordinary* iff it defines a closed point of $X_{[M']}^{\mathbf{v}}$ for some $[M'] \in \mathcal{S}(\mathbf{v})$. The subscheme of non- \mathbf{v} -ordinary points will be denoted by $X_0^{\mathbf{v}}$. We will prove the following Theorem.

Theorem 1.2. *Assume that $(M_{\mathbb{F}'}, \Phi) = (M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}', \Phi)$ is reducible for some finite extension \mathbb{F}' of \mathbb{F} .*

(i) *The subschemes $X_0^{\mathbf{v}}$ and $X_{[M']}^{\mathbf{v}}$ are open and closed in $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \mathbb{F}$ for all isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$.*

(ii) *The scheme $X_0^{\mathbf{v}}$ is connected.*

(iii) *For each $[M'] \in \mathcal{S}(\mathbf{v})$ the scheme $X_{[M']}^{\mathbf{v}}$ is connected. If it is non empty, it is either a single point, or isomorphic to $\mathbb{P}_{\mathbb{F}}^1$.*

(iv) *There are at most two isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$ such that $X_{[M']}^{\mathbf{v}} \neq \emptyset$.*

The structure of the subscheme $X_0^{\mathbf{v}}$ of non- \mathbf{v} -ordinary lattices is much more complicated than in the absolutely simple case. In general $X_0^{\mathbf{v}}$ has many irreducible components of varying dimensions. The main result concerning the irreducible components of $X_0^{\mathbf{v}}$ is the following theorem.

Theorem 1.3. *If $(M_{\mathbb{F}}, \Phi)$ is not isomorphic to the direct sum of two isomorphic one-dimensional ϕ -modules, then the irreducible components of $X_0^{\mathbf{v}}$ are Schubert varieties. Especially they are normal.*

Theorem 1.2 proves a modified version of Kisin's conjecture in the case $k = \mathbb{F}_p$ and $d = 2$, as follows.

For an integer s denote by

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}, s} \subset \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$$

the open and closed subscheme where the rank of the maximal Φ -stable subobject \mathfrak{M}_1 , satisfying $\langle \Phi(\mathfrak{M}_1) \rangle = u^{e-r_1} \mathfrak{M}_1$, is equal to s .

Corollary 1.4. *Assume $p \neq 2$ and let $\rho : G_K \rightarrow V_{\mathbb{F}}$ be any two dimensional continuous representation of G_K . Assume that $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ is a simple algebra for all finite extensions \mathbb{F}' of \mathbb{F} . Then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}, s}$ is geometrically connected for all s . Furthermore*

(i) *If $s = 1$ and $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ for all finite extensions \mathbb{F}' of \mathbb{F} , then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}, s}$ is either empty or a single point.*

If $s = 1$ and $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ for some finite extension \mathbb{F}' of \mathbb{F} , then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}, s}$ is either empty or becomes isomorphic to $\mathbb{P}_{\mathbb{F}'}^1$ after extending the scalars to \mathbb{F}' .

(ii) *If $s = 2$, then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}, s}$ is either empty or a single point.*

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2. SOME NOTATIONS IN THE BUILDING

The method of this paper is to determine all lattices in the building of $GL_2(\bar{\mathbb{F}}((u)))$ that correspond to closed points of $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$. As we know that the scheme we study is a closed reduced subscheme of the affine Grassmannian, we can get information on the structure of $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$ by looking at its closed points.

For the rest of this paper, we fix the following notations: Let $(M_{\mathbb{F}}, \Phi)$ be the object in $\text{Mod}_{\mathbb{F}((u))}^{\phi}$ corresponding to the 2-dimensional Galois representation ρ on $V_{\mathbb{F}}$. Let $\mathbf{v} = (v_{\psi})_{\psi}$ be a collection of integers $v_{\psi} \in \{0, 1, 2\}$ for every $\psi : K \rightarrow \bar{K}_0$. Define

$$(2.1) \quad d' = \sum_{\psi} v_{\psi}.$$

Denote by $\mathbf{r} = \tilde{\mathbf{v}}$ the dual partition, i.e.

$$\begin{aligned} r_1 &= \#\{\psi \mid v_{\psi} \geq 1\} \\ r_2 &= \#\{\psi \mid v_{\psi} \geq 2\}. \end{aligned}$$

Denote by \mathcal{B} the Bruhat-Tits building for $GL_2(\mathbb{F}((u)))$. For any finite extension \mathbb{F}' of \mathbb{F} the building for $GL_2(\mathbb{F}'((u)))$ will be denoted by $\mathcal{B}_{\mathbb{F}'}$. We write

$$\bar{\mathcal{B}} = \bigcup_{\mathbb{F}'/\mathbb{F}} \mathcal{B}_{\mathbb{F}'}$$

for the building for $GL_2(\bar{\mathbb{F}}((u)))$.

We choose an $\mathbb{F}((u))$ -basis e_1, e_2 of $M_{\mathbb{F}}$. Denote by $\mathfrak{M}_0 = \langle e_1, e_2 \rangle$ the standard lattice in the standard apartment \mathcal{A}_0 determined by e_1, e_2 . In this apartment we choose the following coordinates:

Let $(m, n)_0$ denote the lattice $\langle u^m e_1, u^n e_2 \rangle$. Further, we consider another set of coordinates given by $[x, y]_0 = (\frac{x+y}{2}, \frac{y-x}{2})_0$ for $x, y \in \mathbb{Z}$, $x \equiv y \pmod{2}$; i.e. $(m, n)_0 = [m-n, m+n]_0$.

Let $q \in \mathbb{F}((u))^\times$ and set $k = v_u(q) \in \mathbb{Z}$, where v_u is the valuation on $\mathbb{F}((u))$ with $v_u(u) = 1$. The basis $e_1, qe_1 + e_2$ of $M_{\mathbb{F}}$ defines another apartment \mathcal{A}_q which is branching off from the standard apartment at the line defined by $x = k$. Using the Iwasawa decomposition we find

$$\mathcal{B} = \bigcup_{q \in \mathbb{F}((u))} \mathcal{A}_q.$$

For arbitrary $q \in \mathbb{F}((u))$ we choose coordinates in the apartments \mathcal{A}_q , similar to the case of \mathcal{A}_0 . Define

$$(m, n)_q = [m-n, m+n]_q := \langle u^m e_1, u^n (qe_1 + e_2) \rangle \in \mathcal{A}_q.$$

Remark 2.1. (i) The systems of coordinates in the various apartments are compatible in the following sense: For any $x, y \in \mathbb{Z}$, $x \equiv y \pmod{2}$ and $q, q' \in \mathbb{F}((u))$ we have $[x, y]_q = [x, y]_{q'}$ iff $x \leq v_u(q - q')$ which implies

$$[x, y]_q = [x, y]_{q'} \Leftrightarrow [x, y]_q \in \mathcal{A}_q \cap \mathcal{A}_{q'} \Leftrightarrow [x, y]_{q'} \in \mathcal{A}_q \cap \mathcal{A}_{q'}.$$

(ii) We will make use of these coordinates for arbitrary points in the building (not only points corresponding to lattices). We see that $[x, y]_q$ defines a lattice if and only if $x, y \in \mathbb{Z}$ and $x \equiv y \pmod{2}$.

(iii) We extend the above notations in the obvious way to the buildings $\bar{\mathcal{B}}$ and $\mathcal{B}_{\mathbb{F}'}$ for arbitrary finite extensions \mathbb{F}' of \mathbb{F} .

(iv) Two points $[x, y]_q, [x', y']_q \in \mathcal{A}_q$ define the same point in the building for $PGL_2(\mathbb{F}((u)))$ if and only if $x = x'$. Thus the projection from \mathcal{B} onto the building for $PGL_2(\mathbb{F}((u)))$ is given by the projection onto the x -coordinate for every apartment $\mathcal{A}_q \subset \mathcal{B}$.

Definition 2.2. Let \mathfrak{M} and \mathfrak{M}' be lattices in $M_{\mathbb{F}}$. Let a, b be the elementary divisors of \mathfrak{M}' with respect to \mathfrak{M} , i.e. there exists a basis e'_1, e'_2 of \mathfrak{M} such that $\mathfrak{M}' = \langle u^a e'_1, u^b e'_2 \rangle$. Define

$$\begin{aligned} d_1(\mathfrak{M}, \mathfrak{M}') &= |a - b| \\ d_2(\mathfrak{M}, \mathfrak{M}') &= a + b. \end{aligned}$$

Remark 2.3. These quantities have the following meaning in the building:

If $\mathfrak{M} = [x, y]_q$ and $\mathfrak{M}' = [x', y']_{q'}$, then $d_2(\mathfrak{M}, \mathfrak{M}') = y' - y$. Further $d_1(\mathfrak{M}, \mathfrak{M}')$ is the distance between \mathfrak{M} and \mathfrak{M}' in the building for $PGL_2(\mathbb{F}((u)))$. Here, the distance between two lattices joined by an edge is equal to 1. This can be seen as follows: Assume that $\mathfrak{M} = \langle e_1, e_2 \rangle$ is the standard lattice and $\mathfrak{M}' = A\mathfrak{M} = [x, y]_q$, with

$$A = (a_{ij})_{ij} = \begin{pmatrix} u^m & u^n q \\ 0 & u^n \end{pmatrix}.$$

If $a \geq b$ are the elementary divisors of \mathfrak{M}' with respect to \mathfrak{M} , then, by the theory of elementary divisors,

$$d_1(\mathfrak{M}, \mathfrak{M}') = a + b - 2b = m + n - 2 \min_{i,j} v_u(a_{ij}).$$



FIGURE 1. The distance between two lattices in the building for $PGL_2(\mathbb{F}((u)))$ in the cases $x \leq v_u(q)$ and $x \geq v_u(q) \geq 0$.

If $x = m - n \leq v_u(q)$ or $v_u(q) \geq 0$, then $\min_{i,j} v_u(a_{ij}) = \min\{m, n\}$ and hence

$$d_1(\mathfrak{M}, \mathfrak{M}') = a - b = |m - n| = |x|.$$

If $x > v_u(q)$ and $v_u(q) < 0$, then $\min_{i,j} v_u(a_{ij}) = n + v_u(q)$. In this case we find

$$d_1(\mathfrak{M}, \mathfrak{M}') = a - b = m - n - 2v_u(q) = (x - v_u(q)) + (0 - v_u(q)).$$

Compare also Fig. 1 and Fig. 2.

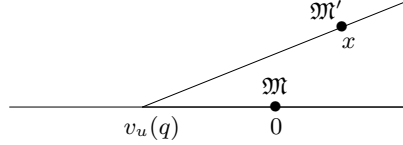


FIGURE 2. The distance between two lattices in the building for $PGL_2(\mathbb{F}((u)))$ in the case $x > v_u(q)$ and $v_u(q) < 0$.

We see that the distance $d_1(\mathfrak{M}, \mathfrak{M}')$ only depends on x, x' (and on $v_u(q - q')$), while $d_2(\mathfrak{M}, \mathfrak{M}')$ only depends on y and y' .

Using this remark, we can extend the distances d_1 and d_2 in an obvious way to the whole building \mathcal{B} (and to $\bar{\mathcal{B}}, \mathcal{B}_{\mathbb{F}'}$). For example

$$d_1([x, y]_q, [0, 0]_0) = \begin{cases} x & \text{if } x \geq 0, v_u(q) \geq 0 \\ -x & \text{if } x < 0, x < v_u(q) \\ x - 2v_u(q) & \text{if } v_u(q) < x, v_u(q) < 0. \end{cases}$$

Lemma 2.4. *Define d' as in (2.1). The closed points $z \in \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}(\bar{\mathbb{F}})$ correspond to the lattices $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ which satisfy*

$$\begin{aligned} d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) &\leq r_1 - r_2 \\ d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) &= 2e - d'. \end{aligned}$$

Proof. If $\mathfrak{M} \subset M_{\bar{\mathbb{F}}}$ is any lattice and if a, b are the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} , then the above conditions read

$$\begin{aligned} a - b &\leq r_1 - r_2 \\ a + b &= 2e - d' = 2e - (r_1 + r_2). \end{aligned}$$

This implies $u^e \mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$. The Jordan type of u on $\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}$ is given by

$$J(u|_{\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}}) = (e - a, e - b).$$

Assuming $a \geq b$ we find:

$$J(u|_{\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}}) \leq \mathbf{r} \Leftrightarrow \begin{cases} b \geq e - r_1 \\ a + b = 2e - d' = 2e - (r_1 + r_2). \end{cases}$$

The lemma follows easily from this. \square

Definition 2.5. A lattice \mathfrak{M} is called *v-admissible* if it satisfies

$$d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) \leq r_1 - r_2 \quad \text{and} \quad d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = 2e - d'.$$

Let \mathfrak{M} be a lattice in $M_{\mathbb{F}}$ and $A \in GL_2(\mathbb{F}((u)))$ be a matrix. We will use the notation $\mathfrak{M} \sim A$ if \mathfrak{M} admits a $\mathbb{F}[[u]]$ -basis b_1, b_2 satisfying $\Phi(b_i) = Ab_i$. Similarly we will use the notation $M_{\mathbb{F}} \sim A$ (use a $\mathbb{F}((u))$ -basis of $M_{\mathbb{F}}$).

Lemma 2.6. For $i = 1, 2$, let $z_i \in \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}(\bar{\mathbb{F}})$ be closed points corresponding to lattices $\mathfrak{M}_i = [x_i, y_i]_{q_i} \in \bar{\mathcal{B}}$. Then $y_1 = y_2$.

Proof. Choose $A, B \in GL_2(\bar{\mathbb{F}}((u)))$ such that $\mathfrak{M}_2 = A\mathfrak{M}_1$ and $\mathfrak{M}_1 \sim B$. Then $\mathfrak{M}_2 \sim \phi(A)BA^{-1}$. Using the theory of elementary divisors it follows that

$$v_u(\det B) = d_2(\mathfrak{M}_i, \langle \Phi(\mathfrak{M}_i) \rangle) = (p-1)v_u(\det A) + v_u(\det B)$$

and hence $v_u(\det A) = 0$ which yields the claim. \square

Definition 2.7. For each $m \in \mathbb{Z}$ define the following subset of $\bar{\mathcal{B}}$:

$$\bar{\mathcal{B}}(m) := \bigcup_{q \in \bar{\mathbb{F}}((u))} \{[x, y]_q \in \mathcal{A}_q \mid y = m\}.$$

Viewing $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}(\bar{\mathbb{F}})$ as a subset of $\bar{\mathcal{B}}$, Lemma 2.6 implies:

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m)$$

for some $m = m(\mathbf{v}) \in \mathbb{Z}$. The subset $\bar{\mathcal{B}}(m)$ is a tree which is (as a topological space) isomorphic to the building for $PGL_2(\bar{\mathbb{F}}((u)))$.

The difference is that not every vertex represents a lattice: A vertex $[x, m]_q \in \bar{\mathcal{B}}(m)$ represents a lattice $\mathfrak{M} \subset M_{\mathbb{F}}$ iff $x \equiv m \pmod{2}$.

Remark 2.8. By construction we have

$$\mathcal{GR}_{V_{\mathbb{F}}, 0} \subset \text{Grass } M_{\mathbb{F}},$$

where $\text{Grass } M_{\mathbb{F}}$ denotes the affine Grassmannian for GL_2 . Since the determinant condition in (1.2) fixes the dimension

$$\dim \langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M} = \sum_{\psi} v_{\psi} = d',$$

the closed subscheme $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$ lies in a connected component of this Grassmannian: If $\mathfrak{M} = A\mathfrak{M}_0$ defines a closed point (where \mathfrak{M}_0 is the standard lattice and A is a matrix), then the valuation of $\det A$ is determined by the dimension of $\langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M}$.

Definition 2.9. For a given collection \mathbf{v} denote by $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}}$ the closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}}, 0}$ consisting of all lattices \mathfrak{M} such that $\dim \langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M} = \sum_{\psi} v_{\psi}$.

Proposition 2.10. If any two of the v_{ψ} differ at most by 1, then

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}} = \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}.$$

Proof. Let \mathfrak{M} be a lattice and $a \geq b$ the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} . Then \mathfrak{M} defines a closed point of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v}}$ if and only if $0 \leq b \leq a \leq e$ and $a + b = 2e - d'$ with $d' = \sum_{\psi} v_{\psi}$. This is equivalent to

$$\begin{aligned} a + b &= 2e - d' \\ a - b &\leq \max\{d', 2e - d'\}. \end{aligned}$$

Indeed, if $a + b = 2e - d'$, then $a - b \leq d'$ is equivalent to $a \leq e$, while $a - b \leq 2e - d'$ is equivalent to $b \geq 0$.

Now if any of the v_{ψ} differ at most by 1 we must have $r_1 = e$ (if all $v_{\psi} \geq 1$) or $r_2 = 0$ (if all $v_{\psi} \leq 1$). In both cases we find $r_1 - r_2 = \max\{d', 2e - d'\}$.

(See also [Ki], Prop. 2.4.6 (4) and [PR1], Thm. B (iii)). \square

3. THE ABSOLUTELY SIMPLE CASE

In this section we will analyze the structure of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ in the case where $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, i.e. for every (finite) extension \mathbb{F}'/\mathbb{F} there is no proper Φ -stable subobject of $(M_{\mathbb{F}'}, \Phi)$.

Lemma 3.1. *If $(M_{\mathbb{F}}, \Phi)$ is absolutely simple, there exists a finite extension \mathbb{F}' of \mathbb{F} , a basis e_1, e_2 of $M_{\mathbb{F}'}$ and $a \in \mathbb{F}'^{\times}$, $s \in \mathbb{Z}$ satisfying*

$$0 \leq s < p^2 - 1 \text{ and } s \not\equiv 0 \pmod{p+1}$$

such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} 0 & au^s \\ 1 & 0 \end{pmatrix}.$$

Proof. This follows from ([Ca], Cor. 8), except that we need to check that $s \not\equiv 0 \pmod{p+1}$. If $p+1|s$, then there would be a proper Φ -stable subspace of $M_{\mathbb{F}'}$ for a quadratic extension \mathbb{F}'' of \mathbb{F}' , namely $\langle \sqrt{a}u^{s/p+1}e_1 + e_2 \rangle \subset M_{\mathbb{F}' \otimes_{\mathbb{F}'} \mathbb{F}''}[\sqrt{a}]$.

The constructions in [Ca] give a basis after extending scalars to the algebraic closure $\bar{\mathbb{F}}$ of \mathbb{F} , but of course this also gives a basis after finite field extension, as there are only finitely many equations to solve. See also ([Im], Lemma 1.2). \square

For the rest of this section we fix the basis e_1, e_2 of Lemma 3.1 as the standard basis of $M_{\bar{\mathbb{F}}}$ and use the coordinates introduced in section 2. Furthermore we fix the point

$$(3.1) \quad P_{\text{irred}} := \left[\frac{s}{p+1}, -\frac{s}{p-1} \right]_0 \in \mathcal{A}_0 \subset \bar{\mathcal{B}}.$$

Proposition 3.2. (i) *The map Φ extends to a map $\bar{\mathcal{B}} \rightarrow \bar{\mathcal{B}}$ also denoted by Φ .*

(ii) *Let $[x, y]_0 \in \mathcal{A}_0$ be any point in the standard apartment. Then*

$$\Phi([x, y]_0) = [-px + s, py + s]_0.$$

(iii) *For any $q \in \bar{\mathbb{F}}((u))^{\times}$ with $k = v_u(q)$ and $[x, y] \in \mathcal{A}_q \setminus \mathcal{A}_0$, i.e. $x > k$, the map Φ is given by*

$$\Phi([x, y]_q) = [px - 2pk + s, py + s]_{q'} \in \mathcal{A}_{q'}$$

for some $q' \in \bar{\mathbb{F}}((u))^{\times}$ with $v_u(q') = -pk + s \neq k$.

(iv) *The point P_{irred} , as defined in (3.1), satisfies $\Phi(P_{\text{irred}}) = P_{\text{irred}}$.*

(v) *If $Q \in \mathcal{B}$ is an arbitrary point, then*

$$d_1(Q, \Phi(Q)) = (p+1)d_1(Q, P_{\text{irred}})$$

$$d_2(Q, \Phi(Q)) = (p-1)d_2(Q, P_{\text{irred}}).$$

Proof. (i) We can use the expressions in (ii) and (iii) to extend Φ .

(ii) We have

$$\begin{aligned}\Phi(u^m e_1) &= u^{pm} \Phi(e_1) = u^{pm} e_2 \\ \Phi(u^n e_2) &= u^{pn} \Phi(e_2) = au^{pn+s} e_1\end{aligned}$$

and hence $\Phi((m, n)_0) = (pn + s, pm)_0$. The statement follows from this.

(iii) We put $v_u(q) = k$ and $\phi(q) = \alpha u^{pk}$ for some $\alpha \in \bar{\mathbb{F}}[[u]]^\times$. If $\mathfrak{M} = (m, n)_q$, then

$$\langle \Phi(\mathfrak{M}) \rangle = \langle u^{pm} e_2, u^{pn} \phi(q) e_2 + au^{pn+s} e_1 \rangle.$$

As $\mathfrak{M} = [m - n, m + n]_q \notin \mathcal{A}_0$ we have $m > n + k$. Hence

$$\begin{aligned}\langle \Phi(\mathfrak{M}) \rangle &= \langle u^{pm} e_2 - \alpha^{-1} u^{p(m-n-k)} (u^{pn} \phi(q) e_2 + au^{pn+s} e_1), u^{pn} \phi(q) e_2 + au^{pn+s} e_1 \rangle \\ &= \langle u^{p(m-k)+s} e_1, u^{p(n+k)} (q' e_1 + e_2) \rangle\end{aligned}$$

with $q' = \alpha^{-1} au^{-pk+s}$. And thus $\Phi((m, n)_q) = (p(m - k) + s, p(n + k))_{q'}$ with $v_u(q') = -pk + s \neq k$, as $k \not\equiv 0 \pmod{p+1}$.

(iv) Obvious.

(v) If $\mathfrak{M} = [x, y]_q$, then the statement on d_2 follows immediately from (ii) and (iii):

$$d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (p - 1)y + s = (p - 1)d_2([x, y]_q, [\frac{s}{p+1}, -\frac{s}{p-1}]_0).$$

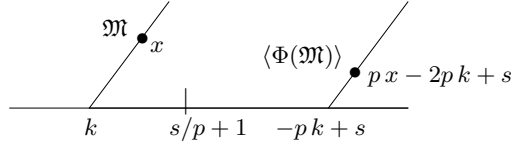


FIGURE 3. The images of \mathfrak{M} and $\Phi(\mathfrak{M})$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$ in the case $\mathfrak{M} \notin \mathcal{A}_0$.

For the statement on d_1 first assume that $\mathfrak{M} = [x, y]_0 \in \mathcal{A}_0$. Then (ii) implies

$$d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = |(p+1)x - s| = (p+1)d_1([x, y]_0, [\frac{s}{p+1}, -\frac{s}{p-1}]_0).$$

If $\mathfrak{M} = [x, y]_q \in \mathcal{A}_q \setminus \mathcal{A}_0$, then $x > k$ and $px - 2pk + s > -pk + s$ which implies $\langle \Phi(\mathfrak{M}) \rangle \notin \mathcal{A}_0$. Now (iii) implies

$$\begin{aligned}d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) &= (x - k) + |-pk + s - k| + (px - 2pk + s - (-pk + s)) \\ &= (p+1)(x - k) + (p+1) \left| k - \frac{s}{p+1} \right| \\ &= \begin{cases} (p+1)(x - \frac{s}{p+1}) & \text{if } k > \frac{s}{p+1} \\ (p+1)(x - k + \frac{s}{p+1} - k) & \text{if } k < \frac{s}{p+1}, \end{cases}\end{aligned}$$

using $k \neq -pk + s$. In both cases the claim follows. (See also Fig. 3) \square

Remark 3.3. The Proposition shows that the absolutely simple case is exactly the case discussed in ([PR2], 6.d, A1). The fixed point in the building is the point P_{irred} and its projection onto the building for $PGL_2(\bar{\mathbb{F}}((u)))$ lies on the edge between two vertices. The set of lattices \mathfrak{M} with $d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) \leq r_1 - r_2$ is identified with a ball around this fixed point.

Let \mathbf{v} be a collection of integers as in the introduction. By Lemma 2.4 and Proposition 3.2(v) we find $\mathcal{GR}_{V_{\mathbb{F},0}}^{\mathbf{v},\text{loc}}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\mathbf{v}))$, where

$$(3.2) \quad m(\mathbf{v}) = (2e - d' - s)/(p - 1).$$

Corollary 3.4. *The scheme $\mathcal{GR}_{V_{\mathbb{F},0}}^{\mathbf{v},\text{loc}}$ is empty if $2e - d' \not\equiv s \pmod{p - 1}$.*

Proof. This follows from Lemma 2.4 and Proposition 3.2. \square

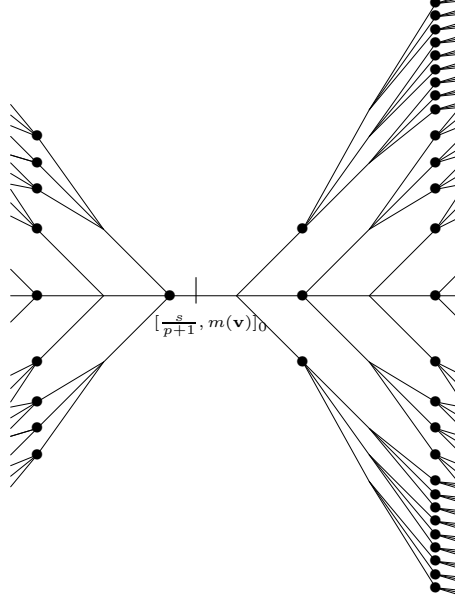


FIGURE 4. This picture illustrates the subset of \mathbf{v} -admissible lattices in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$. This subset is given by all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ satisfying $d_1(\mathfrak{M}, P_{\text{irred}}) \leq (r_1 - r_2)/(p + 1)$. The fat points correspond to \mathbf{v} -admissible lattices.

Now we want to define locally closed subschemes of Grass $M_{\mathbb{F}}$ on which the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} are fixed. Define a function

$$E : \text{Grass } M_{\mathbb{F}} \rightarrow \mathbb{Z}^2.$$

For an extension field L of \mathbb{F} and an L -valued point $z \in (\text{Grass } M_{\mathbb{F}})(L)$ consider the $\mathbb{F}[[u]] \hat{\otimes}_{\mathbb{F}} L$ -lattice \mathfrak{M}_z in $M_{\mathbb{F}} \hat{\otimes}_{\mathbb{F}} L$ corresponding to z . Then $E(z) = (j_1, j_2)$, where $j_1 \geq j_2$ are the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to \mathfrak{M}_z . Recall that there is a partial order on the pairs $(a, b) \in \mathbb{Z}^2$ given by (1.3).

Lemma 3.5. *The function E is lower semi-continuous with respect to the Zariski topology on Grass $M_{\mathbb{F}}$.*

Proof. Let $\eta \rightsquigarrow z$ be a specialization and let \mathfrak{M}_{η} and \mathfrak{M}_z be the lattices corresponding to the points η and z . Denote by $E(\eta) = (a(\eta), b(\eta))$ and $E(z) = (a(z), b(z))$ the elementary divisors of $\langle \Phi(\mathfrak{M}_{\eta}) \rangle$ with respect to \mathfrak{M}_{η} (resp. the elementary divisors of $\langle \Phi(\mathfrak{M}_z) \rangle$ with respect to \mathfrak{M}_z). We mark the specialization by a morphism $f : \text{Spec } R \rightarrow \mathcal{GR}_{V_{\mathbb{F},0}}^{\mathbf{v},\text{loc}}$, where R is a discrete valuation ring with uniformizer t . The

morphism f defines a $R[[u]]$ -lattice \mathfrak{M}_R in $M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} R$. After choosing a basis we find a matrix $C = (c_{ij})_{ij} \in GL_2(R((u))) \cap M_2(R[[u]])$ such that $\mathfrak{M}_R \sim C$. Denote by \bar{c}_{ij} the reduction mod t of the matrix coefficients. Using the theory of elementary divisors we find

$$b(\eta) = \min_{i,j} v_u(c_{ij}) \leq \min_{i,j} v_u(\bar{c}_{ij}) = b(z)$$

and hence $E(\eta) \geq E(z)$ which yields the claim. \square

Definition 3.6. Let $(a, b) \in \mathbb{Z}^2$ such that $a \geq b$. The "*Kisin variety*" associated to (a, b) is

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) = E^{-1}(a, b) \subset \text{Grass } M_{\mathbb{F}}.$$

By Lemma 3.5, this is a locally closed subset and it will be considered as a subscheme with the reduced scheme structure (See also [PR2]).

Now we want to analyze the structure of $\mathcal{G}_{V_{\mathbb{F}}}(a, b)$ and $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\vee, \text{loc}}$. We will make use of the following fact.

Lemma 3.7. *Let b_1, b_2 be any basis of $M_{\mathbb{F}}$. There exists a morphism*

$$\chi : \mathbb{A}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$$

such that $\chi(z) = \langle b_1, zu^{-1}b_1 + b_2 \rangle$ for every closed point $z \in \mathbb{A}_{\mathbb{F}}^1$. The morphism χ extends in a unique way to a morphism

$$\bar{\chi} : \mathbb{P}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}.$$

The image of the point at infinity is given by $\bar{\chi}(\infty) = \langle u^{-1}b_1, ub_2 \rangle$.

Proof. Consider the family

$$\langle b_1, Tu^{-1}b_1 + b_2 \rangle_{\mathbb{F}[T][[u]]} \subset M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}[T]$$

of lattices on $\mathbb{A}^1 = \text{Spec } \mathbb{F}[T]$. This family defines the morphism χ .

Let X be the closed subscheme of $\text{Grass } M_{\mathbb{F}}$ consisting of all lattices \mathfrak{M} that satisfy $u\langle b_1, b_2 \rangle \subset \mathfrak{M} \subset u^{-1}\langle b_1, b_2 \rangle$ and that lie in the same connected component of $\text{Grass } M_{\mathbb{F}}$ as $\langle b_1, b_2 \rangle$. The scheme X is identified with a closed subscheme of the (ordinary) Grassmann variety $\text{Grass}_{\mathbb{F}}(4, 2)$ of 2-dimensional subspaces in \mathbb{F}^4 . The morphism χ factors as follows:

$$\begin{array}{ccc} \mathbb{A}_{\mathbb{F}}^1 & \xrightarrow{\chi} & \text{Grass } M_{\mathbb{F}} \\ & \searrow \chi' & \nearrow \\ & X & \xrightarrow{\iota} \text{Grass}_{\mathbb{F}}(4, 2) \xrightarrow{\iota} \mathbb{P}_{\mathbb{F}}^5 \end{array}$$

where ι is the Plücker embedding. As X is projective, the valuative criterion shows that χ extends in a unique way to \mathbb{P}^1 .

We view $\text{Grass}_{\mathbb{F}}(4, 2)$ as the quotient $GL_{2, \mathbb{F}} \backslash V$, where V is the scheme of 2×4 matrices of rank 2 and $GL_{2, \mathbb{F}}$ acts on V by left multiplication. Now, the computations using Plücker coordinates gives

$$\chi'(z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ z & 0 & 0 & 1 \end{pmatrix}, \quad \iota(\chi'(z)) = (-z : 0 : 0 : 0 : 1 : 0)$$

for all closed points $z \in \mathbb{A}^1(\mathbb{F})$. Hence the extension to \mathbb{P}^1 is

$$(z_1 : z_2) \mapsto (-z_1 : 0 : 0 : 0 : z_2 : 0).$$

The image of the point at infinity is

$$\bar{\chi}((1 : 0)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \mapsto (-1 : 0 : 0 : 0 : 0 : 0).$$

This is the lattice $\langle u^{-1}b_1, ub_2 \rangle$. \square

Remark 3.8. In the building, the \mathbb{F} -valued points of the image of the morphism $\bar{\chi}$ can be illustrated in the following way (if the morphism is defined over \mathbb{F}):

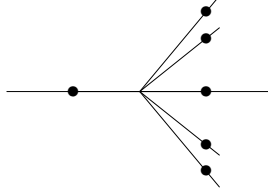


FIGURE 5. The morphism $\bar{\chi}$ in the building for $p = 5$ and $\mathbb{F} = \mathbb{F}_5$.

Similarly, we can define morphisms $\bar{\chi}_1, \bar{\chi}_2 : \mathbb{P}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$ such that

$$\begin{aligned} \text{im}(\bar{\chi}_1) &= \{ \langle u^{n-1}b_1, u^{-(n-1)}(zu^{-1}b_1 + b_2) \rangle \mid z \in \bar{\mathbb{F}} \} \cup \{ \langle u^{-n}b_1, u^n b_2 \rangle \} \\ \text{im}(\bar{\chi}_2) &= \{ \langle u^n b_1, u^{-n}(zb_1 + b_2) \rangle \mid z \in \bar{\mathbb{F}} \} \cup \{ \langle u^{-n}b_1, u^n b_2 \rangle \} \end{aligned}$$

Theorem 3.9. *Assume that $(M_{\mathbb{F}}, \Phi)$ is absolutely simple. Fix a finite extension \mathbb{F}' of \mathbb{F} such that the normal form for Φ of Lemma 3.1 is defined over \mathbb{F}' .*

(a) *For any $(a, b) \in \mathbb{Z}^2$ with $a \geq b$:*

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \neq \emptyset \Leftrightarrow a + b \equiv s \pmod{p-1}, \begin{cases} pa + b \equiv s \pmod{p^2-1} \\ \text{or } pa + b \equiv ps \pmod{p^2-1}. \end{cases}$$

This condition being satisfied, there exists an isomorphism

$$\mathcal{G}_{V_{\mathbb{F}}}(a, b) \otimes_{\mathbb{F}} \mathbb{F}' \cong \mathbb{A}_{\mathbb{F}'}^n,$$

with $n = \lfloor \frac{a-b}{p+1} \rfloor$. Further

$$\overline{\mathcal{G}_{V_{\mathbb{F}}}(a, b)} = \bigcup_{(a', b') \leq (a, b)} \mathcal{G}_{V_{\mathbb{F}}}(a', b').$$

(b) *The scheme $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$ is geometrically connected and irreducible. After extending the scalars to \mathbb{F}' it becomes isomorphic to a Schubert variety in the affine Grassmannian for $M_{\mathbb{F}'} = M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$ with dimension given by*

$$\dim \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \left\lfloor \frac{r_1 - r_2}{p+1} - (-1)^{\epsilon} \frac{s}{p+1} \right\rfloor + \left\lfloor (-1)^{\epsilon} \frac{s}{p+1} \right\rfloor,$$

with $\epsilon = \lfloor \frac{r_1 - r_2}{p+1} \rfloor + \lfloor \frac{s}{p+1} \rfloor + \frac{2e - d' - s}{p-1}$.

[Here as in the rest of the paper, $\lfloor x \rfloor$ denotes the integral part of a real number x .]

Proof. (a) Assume $\mathfrak{M} \in \mathcal{G}_{V_{\mathbb{F}}}(a, b) \neq \emptyset$. Without loss of generality, we may assume $\mathfrak{M} = [x, y]_0 \in \mathcal{A}_0$: if \mathfrak{M} is an arbitrary lattice, then there exists a lattice $\mathfrak{M}' \in \mathcal{A}_0$ such that $d_i(\mathfrak{M}, P_{\text{irred}}) = d_i(\mathfrak{M}', P_{\text{irred}})$ for $i = 1, 2$ (compare Fig. 4, for example). By Lemma 3.2(v) and Definition 2.2, the condition for $\mathfrak{M} = [x, y]_0 \in \mathcal{G}_{V_{\mathbb{F}}}(a, b)$ is

$$(p+1)d_1(\mathfrak{M}, P_{\text{irred}}) = a - b \quad (p-1)d_2(\mathfrak{M}, P_{\text{irred}}) = a + b.$$

By an explicit computation of these distances, this is equivalent to

$$\left| x - \frac{s}{p+1} \right| = \frac{a-b}{p+1} \quad y + \frac{s}{p-1} = \frac{a+b}{p-1}.$$

The second equation gives $s \equiv a+b \pmod{p-1}$ and the sum of both equations gives $s \equiv pa+b \pmod{p^2-1}$ if $(p+1)x > s$ and $ps \equiv pa+b \pmod{p^2-1}$ if $(p+1)x < s$ (using the fact that $x+y$ and $x-y$ are even).

Conversely, suppose $s \equiv a+b \pmod{p-1}$ and $s \equiv pa+b \pmod{p^2-1}$ and define

$$x = \frac{a-b+s}{p+1} \quad y = \frac{a+b-s}{p-1}.$$

Then we have $y \in \mathbb{Z}$ and $x+y \in 2\mathbb{Z}$. Thus $[x, y]_0$ defines a lattice $\mathfrak{M} \in \mathcal{G}_{V_F}(a, b)$. If $ps \equiv pa+b \pmod{p^2-1}$ we use

$$x = \frac{s-(a-b)}{p+1} \quad y = \frac{a+b-s}{p-1}.$$

Now fix the sum $a+b$ and denote by y the integer solving the equation

$$(p-1)y + s = a+b.$$

Let us assume that $x_0 := \lfloor \frac{s}{p+1} \rfloor \equiv y \pmod{2}$ (the case $x_0 \not\equiv y \pmod{2}$ admits a similar treatment). In this case $[x_0, y]_0$ defines a lattice \mathfrak{M}_0 and we denote by X the connected component of $\text{Grass } M_{\mathbb{F}'}$ containing \mathfrak{M}_0 , i.e. $X(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(y)\}$. For each $m \geq 0$, there is a morphism $f_m : \mathbb{A}_{\mathbb{F}'}^{2m+1} \rightarrow X$ given by the family of lattices

$$\left\langle u^{(x_0+y)/2} u^{m+1} e_1, u^{(y-x_0)/2} u^{-(m+1)} \left(\sum_{i=1}^{2m+1} T_i u^{i+x_0} e_1 + e_2 \right) \right\rangle \\ \subset M_{\mathbb{F}'} \hat{\otimes}_{\mathbb{F}'} (\mathbb{F}'[T_1, \dots, T_{2m+1}])$$

on $\mathbb{A}_{\mathbb{F}'}^{2m+1} = \text{Spec } \mathbb{F}'[T_1, \dots, T_{2m+1}]$. Let $V_m \cong \mathbb{A}_{\mathbb{F}'}^{2m+1}$ be its image. We have

$$(3.3) \quad \bar{\mathcal{B}}(y) \supset V_m(\bar{\mathbb{F}}) = \{ \langle u^{(x_0+y)/2} u^{m+1} e_1, u^{(y-x_0)/2} u^{-(m+1)} (q e_1 + e_2) \rangle \mid q = \sum_{i=1}^{2m+1} a_i u^{i+x_0} \},$$

with $a_1 \dots a_{2m+1} \in \bar{\mathbb{F}}$. Similarly, define for $m \geq 0$ a morphism $g_m : \mathbb{A}_{\mathbb{F}'}^{2m} \rightarrow X$ given by the family of lattices

$$\left\langle u^{(x_0+y)/2} u^{-m} \left(e_1 + \sum_{i=0}^{2m-1} T_i u^{i-x_0} e_2 \right), u^{(y-x_0)/2} u^m e_2 \right\rangle \\ \subset M_{\mathbb{F}'} \hat{\otimes}_{\mathbb{F}'} (\mathbb{F}'[T_0, \dots, T_{2m-1}])$$

and let $U_m \cong \mathbb{A}_{\mathbb{F}'}^{2m}$ be its image. We have

$$(3.4) \quad \bar{\mathcal{B}}(y) \supset U_m(\bar{\mathbb{F}}) = \{ \langle u^{(x_0+y)/2} u^{-m} (e_1 + q e_2), u^{(y-x_0)/2} u^m e_2 \rangle \mid q = \sum_{i=0}^{2m-1} a_i u^{i-x_0} \},$$

with $a_0 \dots a_{2m-1} \in \bar{\mathbb{F}}$. It is easy to see that every lattice $\mathfrak{M} \in \bar{\mathcal{B}}(y)$ is either of the form (3.3) or of the form (3.4) for some $m \geq 0$. Thus

$$X = \left(\bigcup_{m \geq 0} V_m \right) \cup \left(\bigcup_{m \geq 0} U_m \right).$$

We claim

$$(3.5) \quad \mathfrak{M} \in V_m(\bar{\mathbb{F}}) \implies d_1(\mathfrak{M}, P_{\text{irred}}) = 2m + 2 - \xi$$

$$(3.6) \quad \mathfrak{M} \in U_m(\bar{\mathbb{F}}) \implies d_1(\mathfrak{M}, P_{\text{irred}}) = 2m + \xi,$$

where $\xi = \frac{s}{p+1} - x_0$ denotes the fractional part of $\frac{s}{p+1}$.

Indeed, if $\mathfrak{M} \in V_m(\bar{\mathbb{F}})$, then $\mathfrak{M} = [x_0 + 2m + 2, y]_q$ for some $q \in \bar{\mathbb{F}}((u))$ with $v_u(q) > x_0$ and hence

$$d_1(\mathfrak{M}, P_{\text{irred}}) = x_0 + 2m + 2 - \frac{s}{p+1} = 2m + 2 - \xi.$$

The statement on U_m follows by a more complicated computation or by a symmetry argument: The choice of apartments \mathcal{A}_q and coordinates $[-, -]_q$ depends on the order of e_1 and e_2 . Interchanging e_1 and e_2 yields expressions for the lattices $\mathfrak{M} \in U_m$ similar to the above expressions for V_m (if $\mathfrak{M} \in U_m$ is a lattice, then $\mathfrak{M} = [-x_0 + 2m, y]_q$ for some q) while it maps the point P_{irred} to $[-\frac{s}{p+1}, -\frac{s}{p-1}]_0$ and hence the claim follows by the same computation.

Now equation (3.5) and (3.6) together with Proposition 3.2 (v) imply

$$(3.7) \quad \begin{aligned} V_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\bar{\mathbb{F}}}}(a_{\text{odd}}(m), b_{\text{odd}}(m))(\bar{\mathbb{F}}) \\ U_m(\bar{\mathbb{F}}) &\subset \mathcal{G}_{V_{\bar{\mathbb{F}}}}(a_{\text{even}}(m), b_{\text{even}}(m))(\bar{\mathbb{F}}) \end{aligned}$$

for some $(a_{\text{odd}}(m), b_{\text{odd}}(m)), (a_{\text{even}}(m), b_{\text{even}}(m)) \in \mathbb{Z}^2$ with

$$(3.8) \quad \begin{aligned} a_{\text{odd}}(m) + b_{\text{odd}}(m) &= a_{\text{even}}(m') + b_{\text{even}}(m') = (p-1)y + s \\ a_{\text{odd}}(m) - b_{\text{odd}}(m) &= (p+1)(2m+2-\xi) \\ a_{\text{even}}(m) - b_{\text{even}}(m) &= (p+1)(2m+\xi) \end{aligned}$$

and $0 < \xi < 1$ implies that all these pairs are pairwise distinct when m runs over all positive integers.

As U_m and V_m cover X , the inclusions in (3.7) are actually equalities. Furthermore $V_m = \mathcal{G}_{V_{\bar{\mathbb{F}}}}(a_{\text{odd}}(m), b_{\text{odd}}(m))$ as schemes, as both are reduced locally closed subschemes of Grass $M_{\bar{\mathbb{F}}'}$ with the same underlying point set. Finally (3.8) yields

$$\dim V_m = 2m + 1 = \lfloor 2m + 2 - \xi \rfloor = \lfloor \frac{a_{\text{odd}}(m) - b_{\text{odd}}(m)}{p+1} \rfloor.$$

The conclusion for U_m is similar.

To finish the proof of (a), it remains to show that $U_m \subset \overline{V_m}$ and $V_{m-1} \subset \overline{U_m}$. We will prove the first assertion: the second is proved in the same way.

Let $z_1 \in U_m$ be an arbitrary point corresponding to a lattice

$$\mathfrak{M}_1 = \langle u^{(x_0+y)/2} u^{-m} (e_1 + qe_2), u^{(y-x_0)/2} u^m e_2 \rangle$$

with $q = \sum_{i=0}^{2m-1} a_i u^{i-x_0}$ and let $z_2 \in V_m$ be the point corresponding to

$$\mathfrak{M}_2 = \langle u^{(x_0+y)/2} u^{m+1} e_1, u^{(y-x_0)/2} u^{-(m+1)} e_2 \rangle.$$

There exists a basis b_1 and b_2 of $M_{\bar{\mathbb{F}}}$ such that

$$\begin{aligned} \langle b_1, b_2 \rangle &= \mathfrak{M}_0 = [x_0, y]_0, \\ \langle u^{-m} b_1, u^m b_2 \rangle &= \mathfrak{M}_1, \\ \langle u^{m+1} b_1, u^{-(m+1)} b_2 \rangle &= \mathfrak{M}_2. \end{aligned}$$

Explicitly, we may choose

$$b_1 = u^{(x_0+y)/2} (e_1 + qe_2), \quad b_2 = u^{(y-x_0)/2} e_2.$$

Applying Lemma 3.7 (resp. Remark 3.8) with the basis $ub_1, u^{-1}b_2$, we obtain a morphism $\chi : \mathbb{A}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$ that is given by $\chi(z) = \langle u^{m+1}b_1, u^{-(m+1)}(zub_1 + b_2) \rangle$ on closed points and we easily find $\text{im } \chi \subset V_m \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$. As $\overline{V_m} \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ is projective, the morphism χ extends to a morphism from \mathbb{P}^1 to $\overline{V_m} \otimes_{\mathbb{F}'} \bar{\mathbb{F}}$ and the point at infinity is mapped to z_1 (Fig. 6 illustrates the image of the morphism $\bar{\chi}$ in the building). The fat points are the lattices in the image of $\bar{\chi}$. Hence $z_1 \in \overline{V_m}(\mathbb{F})$ and the claim follows.

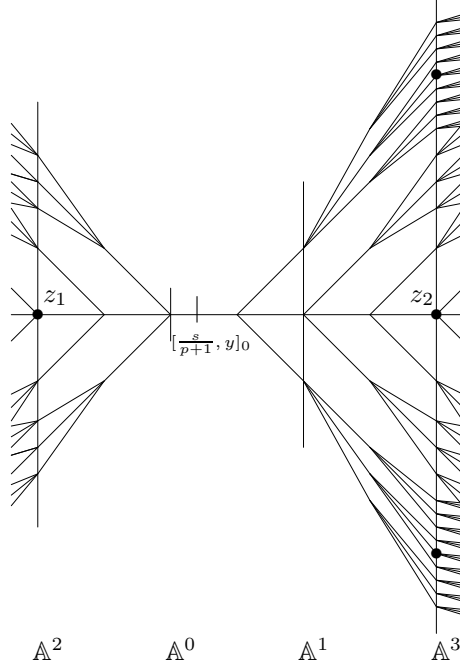


FIGURE 6. The stratification with affine spaces in the building. Fat points mark the image of an exemplary morphism $\bar{\chi}$.

(b) For a given collection \mathbf{v} we have

$$(3.9) \quad \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \bigcup_{\substack{a+b=2e-d' \\ e-r_1 \leq b \leq a \leq e-r_2}} \mathcal{G}_{V_{\mathbb{F}}}(a, b),$$

where d' is the integer defined in (2.1). Hence the scheme is geometrically irreducible, because the restriction of the order " \leq " on the pairs

$$\{(a, b) \in \mathbb{Z}^2 \mid a + b = m(\mathbf{v})\},$$

where $m(\mathbf{v})$ is given by (3.2), is a total order. Of course this also implies connectedness.

The dimension of $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$ is given by the dimension of the maximal affine space in (3.9). We assume that ϵ is even, i.e. $\lfloor \frac{r_1-r_2}{p+1} \rfloor + x_0 \equiv m(\mathbf{v}) \pmod{2}$. The computations in the other case are similar.

In this case the affine subspace of maximal dimension consists of all lattices

$\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 - \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}).$$

(if the latter distance is $\leq \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if $d_1([x_0 - \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}) > \frac{r_1-r_2}{p+1}$). Hence its dimension is either $n := \lfloor \frac{r_1-r_2}{p+1} \rfloor$ (in the first case) or $n-1$ (in the second case). This yields the claim on the dimension:

$$\begin{aligned} \dim \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} &= \lfloor \frac{r_1-r_2}{p+1} - \frac{s}{p+1} \rfloor + \lfloor \frac{s}{p+1} \rfloor \\ &= \begin{cases} n & \text{if } \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor \leq \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \\ n-1 & \text{if } \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor > \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor. \end{cases} \end{aligned}$$

We further see that the set of \mathbf{v} -admissible lattices is exactly the set of lattices in $\bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$(3.10) \quad \begin{aligned} d_1(\mathfrak{M}, [x_0, m(\mathbf{v})]_0) &\leq n & \text{if } x_0 + \lfloor \frac{r_1-r_2}{p+1} \rfloor - \frac{s}{p+1} \leq \frac{r_1-r_2}{p+1} \\ d_1(\mathfrak{M}, [x_0 + 1, m(\mathbf{v})]_0) &\leq n-1 & \text{otherwise} \end{aligned}$$

and hence this is the set of lattices whose elementary divisors (a, b) with respect to a lattice \mathfrak{N} satisfy $(a, b) \leq (a_{\max}, b_{\max})$ for some given integers a_{\max}, b_{\max} . For \mathfrak{N} we choose one of the lattices

$$[x_0, m(\mathbf{v})]_0, [x_0, m(\mathbf{v}) - 1]_0 \quad \text{or} \quad [x_0 + 1, m(\mathbf{v})]_0, [x_0 + 1, m(\mathbf{v}) - 1]_0$$

depending on the cases as listed in (3.10) and on $x_0 - m(\mathbf{v}) \pmod 2$. Since we know that $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is reduced, we find that it is isomorphic to a Schubert variety in the affine Grassmannian after extending the scalars to \mathbb{F}' .

If ϵ is odd, then the maximal affine subspace consists of all lattices $\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v}))$ with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + 1 + \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if the latter distance is $\leq \frac{r_1-r_2}{p+1}$) or of the lattices with

$$d_1(\mathfrak{M}, P_{\text{irred}}) = d_1([x_0 + 1 - \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}})$$

(if $d_1([x_0 + 1 + \lfloor \frac{r_1-r_2}{p+1} \rfloor, m(\mathbf{v})]_0, P_{\text{irred}}) > \frac{r_1-r_2}{p+1}$). We find

$$\begin{aligned} \dim \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} &= \lfloor \frac{r_1-r_2}{p+1} + \frac{s}{p+1} \rfloor + \lfloor -\frac{s}{p+1} \rfloor \\ &= \begin{cases} n & \text{if } 1 - (\frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor) \leq \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \\ n-1 & \text{if } 1 - (\frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor) > \frac{r_1-r_2}{p+1} - \lfloor \frac{r_1-r_2}{p+1} \rfloor \end{cases} \end{aligned}$$

and the conclusion for the isomorphism with a Schubert variety is similar. \square

As a consequence of the theorem, we may determine the cases when $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is a single point.

Corollary 3.10. *Denote by $\xi = \frac{s}{p+1} - \lfloor \frac{s}{p+1} \rfloor$ the fractional part of $\frac{s}{p+1}$.*

$$\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} = \{*\} \Leftrightarrow \begin{cases} 0 + \xi \leq \frac{r_1-r_2}{p+1} < 2 - \xi & \text{if } \lfloor \frac{s}{p+1} \rfloor \equiv \frac{2e-d'-s}{p-1} \pmod 2 \\ 1 - \xi \leq \frac{r_1-r_2}{p+1} < 1 + \xi & \text{if } \lfloor \frac{s}{p+1} \rfloor \not\equiv \frac{2e-d'-s}{p-1} \pmod 2. \end{cases}$$

Proof. This is just the case where the dimension of $\mathcal{GR}_{V_F,0}^{\mathbf{v},\text{loc}}$ is zero. More explicitly: If $x_0 = \lfloor \frac{s}{p+1} \rfloor \equiv m(\mathbf{v}) \pmod{2}$, then $[x_0, m(\mathbf{v})]$ is the unique lattice with minimal distance d_1 from P_{irred} . We have

$$d_1([x_0, m(\mathbf{v})]_0, P_{\text{irred}}) = \xi.$$

Thus this lattice is \mathbf{v} -admissible if and only if $\frac{r_1-r_2}{p+1} \geq \xi$. There is no other \mathbf{v} -admissible lattice iff the lattices \mathfrak{M} with $d_1(\mathfrak{M}, P_{\text{irred}}) = 2 - \xi$ are not \mathbf{v} -admissible. This yields the claim.

The case $x_0 \not\equiv m(\mathbf{v}) \pmod{2}$ is similar. Instead of $[x_0, m(\mathbf{v})]_0$ we have to consider the lattice $[x_0 + 1, m(\mathbf{v})]_0$. \square

4. THE REDUCIBLE CASE

In this section we want to analyze the case, where $(M_{\mathbb{F}}, \Phi)$ admits a proper Φ -stable subobject, at least after extending the scalars to some finite extension of \mathbb{F} . Before we start to determine the set of \mathbf{v} -admissible lattices in the building, we want to formulate the precise statement on the connected components of $\mathcal{GR}_{V_F,0}^{\mathbf{v},\text{loc}}$. We first define some open and closed subschemes of $\mathcal{GR}_{V_F,0}^{\mathbf{v},\text{loc}}$.

Definition 4.1. For $a \in \bar{\mathbb{F}}^\times$ and $j \in \mathbb{Z}_{\geq 0}$ define $(\mathfrak{M}^j(a), \Phi_a^j)$ by

$$\mathfrak{M}^j(a) = \bar{\mathbb{F}}[[u]] , \quad \Phi_a^j(1) = au^j.$$

Definition 4.2. A \mathbf{v} -admissible lattice $\mathfrak{M} \subset M_{\mathbb{F}}$ is called *\mathbf{v} -ordinary* if there exists a short exact sequence

$$(4.0.1) \quad 0 \rightarrow (\mathfrak{M}^{e-r_1}(a), \Phi_a^{e-r_1}) \rightarrow (\mathfrak{M}, \Phi) \rightarrow (\mathfrak{M}^{e-r_2}(b), \Phi_b^{e-r_2}) \rightarrow 0$$

for some $a, b \in \bar{\mathbb{F}}^\times$.

Remark 4.3. The determinant condition in (1.2) implies that

$$(4.0.2) \quad u^{e-r_1}\mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset u^{e-r_2}\mathfrak{M}$$

for all \mathbf{v} -admissible lattices \mathfrak{M} . Hence the \mathbf{v} -ordinary lattices are the lattices which admit a Φ -stable subobject with the minimal possible elementary divisors. If a \mathbf{v} -admissible lattice (\mathfrak{M}, Φ) admits a subobject isomorphic to $(\mathfrak{M}^{e-r_1}(a), \Phi_a^{e-r_1})$ for some $a \in \bar{\mathbb{F}}^\times$, then the quotient has no u -torsion by (4.0.2) and is isomorphic to $(\mathfrak{M}^{e-r_2}(b), \Phi_b^{e-r_2})$ for some $b \in \bar{\mathbb{F}}^\times$, because the sum of the elementary divisors is fixed by (1.2). Hence (\mathfrak{M}, Φ) is \mathbf{v} -ordinary in this case.

Denote by $\mathcal{S}(\mathbf{v})$ the set of isomorphism classes of one dimensional $\bar{\mathbb{F}}((u))$ -modules M' with ϕ -linear map $\Phi' \neq 0$ such that M' admits a (unique) lattice $\mathfrak{M}_{[M']} \subset M'$ with $\langle \Phi(\mathfrak{M}_{[M']}) \rangle = u^{e-r_1}\mathfrak{M}_{[M']}$. The elements of $\mathcal{S}(\mathbf{v})$ are in bijection with the elements of $\bar{\mathbb{F}}^\times$: For each $a \in \bar{\mathbb{F}}^\times$ there is a unique isomorphism class represented by

$$(4.0.3) \quad (M_a, \Phi_a) = (\mathfrak{M}^{e-r_1}(a)[\frac{1}{u}], \Phi_a^{e-r_1}).$$

Set $X = \mathcal{GR}_{V_F,0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. On X there is a universal sheaf of $\bar{\mathbb{F}}[[u]] \hat{\otimes}_{\mathbb{F}} \mathcal{O}_X = \mathcal{O}_X[[u]]$ -lattices $\mathcal{M} \subset M_{\bar{\mathbb{F}}} \hat{\otimes}_{\mathbb{F}} \mathcal{O}_X$ satisfying

$$u^e \mathcal{M} \subset (\text{id} \otimes \Phi) \phi^* \mathcal{M} \subset \mathcal{M}.$$

For each $[M'] \in \mathcal{S}(\mathbf{v})$ define a sheaf of \mathcal{O}_X -modules

$$(4.0.4) \quad \mathcal{F}_{[M']} = \mathcal{H}om_{\mathcal{O}_X[[u]], \Phi}(\mathfrak{M}_{[M']} \hat{\otimes}_{\mathbb{F}} \mathcal{O}_X, \mathcal{M})$$

where the subscript Φ indicates that the homomorphism have to commute with the semi-linear maps that are part of the data.

Proposition 4.4. (i) For each $[M'] \in \mathcal{S}(\mathbf{v})$ the sheaf $\mathcal{F}_{[M']}$ is a coherent \mathcal{O}_X -module.

(ii) A closed point $x \in \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ corresponds to a non- \mathbf{v} -ordinary lattice if and only if $\mathcal{F}_{[M']} \otimes \kappa(x) = 0$ for all $[M'] \in \mathcal{S}(\mathbf{v})$.

Proof. (i) For the isomorphism class $[M']$ we choose a representative of the form M_a defined in (4.0.3). Let $U = \text{Spec } A \subset X$ an affine open. We claim

(a) $\text{Hom}_{A[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} A, \mathcal{M}(U))$ is a finitely generated A -module.

(b) If $V = \text{Spec } B \subset U$ is an affine open we have

$$(4.0.5) \quad \text{Hom}_{B[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} B, \mathcal{M}(V)) \cong \text{Hom}_{A[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} A, \mathcal{M}(U)) \otimes_A B.$$

This implies the first part of the Proposition.

Proof of (a): Because $\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} A$ is a free $A[[u]]$ -module of rank one, a morphism is given by the image of 1 and hence

$$\text{Hom}_{A[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} A, \mathcal{M}(U)) \cong N_A \subset \mathcal{M}(U),$$

where N_A is the A -submodule of all $v \in \mathcal{M}(U)$ satisfying $\Phi(v) = au^{e-r_1}v$. We claim that the reduction modulo u^{e+1} induces an injective homomorphism

$$N_A \hookrightarrow \mathcal{M}(U)/u^{e+1}\mathcal{M}(U)$$

and hence N_A is finitely generated as an A -module, because the scheme X is noetherian. Now, if $0 \neq v = u^n w \in N_A$ with $n \geq 0$ and $w \in \mathcal{M}(U) \setminus u\mathcal{M}(U)$, then

$$u^{pn}\Phi(w) = \Phi(u^n w) = au^{e-r_1+n}w$$

and hence $0 \leq e - r_1 - (p-1)n \leq e$ which implies $n \leq e$.

Proof of (b): We have the following commutative diagram

$$\begin{array}{ccccc} \text{Hom}_{A[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} A, \mathcal{M}(U)) & \xrightarrow{\cong} & N_A & \hookrightarrow & \mathcal{M}(U) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_{B[[u],\Phi]}(\mathfrak{M}_{[M']} \widehat{\otimes}_{\mathbb{F}} B, \mathcal{M}(V)) & \xrightarrow{\cong} & N_B & \hookrightarrow & \mathcal{M}(V) \cong \mathcal{M}(U) \widehat{\otimes}_A B \end{array}$$

As N_A is a finitely generated A -module, we do not need to complete the tensor product to obtain N_B from N_A (there are only finitely many denominators). Hence (4.0.5) is an isomorphism.

(ii) Let $[M'] \in \mathcal{S}(\mathbf{v})$ be an isomorphism class and suppose that $x \in \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ is a closed point corresponding to a lattice \mathfrak{M} such that $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, i.e. there exists a non trivial morphism

$$f : \mathfrak{M}_{[M']} \rightarrow \mathfrak{M}.$$

As both sides are free $\mathbb{F}[[u]]$ -modules and the morphism is non trivial, it is injective. We have to convince ourselves that $\text{coker } f$ has no u -torsion: in this case \mathfrak{M} is the extension of free $\mathbb{F}[[u]]$ -modules of rank 1 (an extension of $\text{coker } f$ by $\text{im } f$), and hence \mathbf{v} -ordinary.

We write $f(1) = u^n v$ for some $n \in \mathbb{Z}$ and $v \in \mathfrak{M} \setminus u\mathfrak{M}$ and claim $n = 0$. Because of $\Phi(f(1)) = f(\Phi(1))$ we find $\Phi(v) = au^{e-r_1-(p-1)n}v \in \Phi(\mathfrak{M}) \subset u^{e-r_1}\mathfrak{M}$ for some $a \in \mathbb{F}^\times$ and hence $n = 0$.

Conversely, if \mathfrak{M} is \mathbf{v} -ordinary, then the inclusion of the Φ -stable subobject defines a nontrivial morphism $\mathfrak{M}_{[M']} \rightarrow \mathfrak{M}$ for some $[M'] \in \mathcal{S}(\mathbf{v})$. \square

Definition 4.5. For each isomorphism class $[M'] \in \mathcal{S}(\mathbf{v})$ define

$$X_{[M']}^{\mathbf{v}} = \{x \in \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \mid \mathcal{F}_{[M']} \otimes \kappa(x) \neq 0\}.$$

Further define

$$X_0^{\mathbf{v}} = \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \setminus \bigcup_{[M'] \in \mathcal{S}(\mathbf{v})} X_{[M']}^{\mathbf{v}}.$$

By the Proposition below these subsets are open and closed and hence they come along with a canonical scheme structure.

Proposition 4.6. (i) The subset $X_{[M']}^{\mathbf{v}}$ is open and closed for each $[M'] \in \mathcal{S}(\mathbf{v})$.
(ii) The subset $X_0^{\mathbf{v}}$ is open and closed.

Proof. (i) It is clear that $X_{[M']}^{\mathbf{v}}$ is closed, as $\mathcal{F}_{[M']}$ is coherent. We show that it is closed under cospecialization.

Let $\eta \rightsquigarrow x$ be a specialization with $x \in X_{[M']}^{\mathbf{v}}$ and assume that x is a closed point. We mark this specialization by $\text{Spec } R \rightarrow X$, where R is a discrete valuation ring with uniformizer t and residue field $\bar{\mathbb{F}}$. Denote by \mathfrak{M}_R the $R[[u]]$ -lattice in $M_{\bar{\mathbb{F}} \otimes_{\mathbb{F}} R}$ defined by this morphism. Because of $\mathcal{F}_{[M']} \otimes \kappa(x) \neq 0$, there is a non trivial morphism $\mathfrak{M}_{[M']} \rightarrow \mathfrak{M}_x$ and hence there is a basis vector $b_1 \in M_{\bar{\mathbb{F}}}$ such that $\Phi(b_1) = au^{e-r_1}b_1$ for some $a \in \bar{\mathbb{F}}^\times$. As \mathfrak{M}_R is a free $R[[u]]$ -module, there is a basis of \mathfrak{M}_R such that

$$\mathfrak{M}_R \sim \begin{pmatrix} \alpha & \gamma \\ 0 & \delta \end{pmatrix},$$

for $\alpha, \gamma, \delta \in R[[u]]$, with $\alpha \equiv au^{e-r_1} \pmod{t}$. But the determinant condition in (1.2) implies $v_u(\alpha) \geq e - r_1$. Hence $v_u(\alpha) = e - r_1$ and $\eta \in X_{[M'']}^{\mathbf{v}}$ for some $[M''] \in \mathcal{S}(\mathbf{v})$.

If $[M'] = [M'']$ we are done.

Assume $[M'] \neq [M'']$. As $X_{[M']}^{\mathbf{v}}$ is closed, we have $x \in X_{[M']}^{\mathbf{v}} \cap X_{[M'']}^{\mathbf{v}}$. In this case \mathfrak{M}_x admits two linear independent subspaces:

$$\mathfrak{M}_x \sim \begin{pmatrix} au^{e-r_1} & 0 \\ 0 & bu^{e-r_1} \end{pmatrix}$$

and hence $e - r_1 = e - r_2$. Now we easily deduce $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} = \{\mathfrak{M}_x\}$ and the claim follows.

(ii) This follows from the first part of the Proposition together with the fact that the one-dimensional Φ -invariant subspaces of $M_{\bar{\mathbb{F}}}$ which admit an integral model \mathfrak{M} with $\langle \Phi(\mathfrak{M}) \rangle = u^{e-r_1}\mathfrak{M}$ run over a finite set of isomorphism classes of one-dimensional objects:

Assume that there are two different one-dimensional Φ -stable subspaces $\langle b_1 \rangle$ and $\langle b_2 \rangle$ of $M_{\bar{\mathbb{F}}}$ such that $\Phi(b_i) = a_i u^{e-r_1} b_i$, for $i = 1, 2$. Then b_1 and b_2 are linear independent.

If $a_1 \neq a_2$, then $\langle b_1 + qb_2 \rangle$ is not Φ -stable for all $q \in \bar{\mathbb{F}}((u))^\times$ and hence there are only two isomorphism classes.

If $a_1 = a_2$, then there is a unique such isomorphism class given by $[M_a]$. \square

We will see below that the open and closed subschemes $X_{[M']}^{\mathbf{v}}$ and $X_0^{\mathbf{v}}$ of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ are connected and hence turn out to be the connected components of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$.

Now we want to determine the subset of \mathbf{v} -admissible lattices in the building. As we are assuming that $(M_{\mathbb{F}}, \Phi)$ is reducible, at least after extending scalars, there exists a finite extension \mathbb{F}' of \mathbb{F} and a basis e_1, e_2 of $M_{\mathbb{F}'} = M_{\mathbb{F}} \widehat{\otimes}_{\mathbb{F}} \mathbb{F}'$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

for some $a, b \in \mathbb{F}'^\times$, $\gamma \in \mathbb{F}'((u))$ and $s, t \in \mathbb{Z}$ with $0 \leq s, t < p-1$. We choose this basis to be the standard basis.

Lemma 4.7. (i) *The map Φ extends to a map $\bar{\mathcal{B}} \rightarrow \bar{\mathcal{B}}$ also denoted by Φ .*

(ii) *For $q \in \mathbb{F}'((u))$ and $[x, y]_q \in \mathcal{A}_q$ the map Φ is given by*

$$\Phi([x, y]_q) = [px + s - t, py + s + t]_{q'}$$

with $q' = b^{-1}u^{-t}(au^s\phi(q) + \gamma)$.

Proof. (i) We can use the expressions in (ii) to extend Φ .

(ii) We have $\Phi(u^m e_1) = au^{pm+s}e_1$ and

$$\begin{aligned} \Phi(u^n(qe_1 + e_2)) &= u^{pn}(au^s\phi(q)e_1 + \gamma e_1 + bu^t e_2) \\ &= bu^{pn+t}(b^{-1}u^{-t}(\gamma + au^s\phi(q))e_1 + e_2). \end{aligned}$$

The Lemma follows from this. \square

Corollary 4.8. *The scheme $\mathcal{GR}_{V_{\mathbb{F},0}}^{\mathbf{v},\text{loc}}$ is empty if $2e - d' \not\equiv s + t \pmod{p-1}$.*

Proof. This follows from Lemma 2.4 and Lemma 4.7: We have

$$d_2([x, y]_q, \Phi([x, y]_q)) = (p-1)y + s + t$$

and this distance must be equal to $2e - d'$ if $[x, y]_q$ is \mathbf{v} -admissible. \square

We assume that the scheme is non empty and define

$$(4.0.6) \quad P_{\text{red}} = \left[\frac{t-s}{p-1}, -\frac{t+s}{p-1} \right] \in \mathcal{A}_0 \subset \bar{\mathcal{B}}$$

$$(4.0.7) \quad m(\mathbf{v}) = \frac{2e-d'-(s+t)}{p-1} \in \mathbb{Z}.$$

These definitions imply $\mathcal{GR}_{V_{\mathbb{F},0}}^{\mathbf{v},\text{loc}}(\bar{\mathbb{F}}) \subset \bar{\mathcal{B}}(m(\mathbf{v}))$.

There are three different cases which we have to study in order to determine the set of \mathbf{v} -admissible lattices. It makes a difference whether $(M_{\bar{\mathbb{F}}}, \Phi)$ is a split or a non-split extension of two one-dimensional objects. In the split case there are two possibilities: Either the direct summands are isomorphic or non-isomorphic.

4.1. The case $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_1, \Phi_1)$. In this section we want to analyze the case where $(M_{\bar{\mathbb{F}}}, \Phi)$ becomes isomorphic to a direct sum of two isomorphic one-dimensional objects after possibly extending the scalars to some finite extension of \mathbb{F} , i.e. we want to assume that there exists \mathbb{F}'/\mathbb{F} and an $\mathbb{F}'((u))$ -basis e_1, e_2 of $M_{\mathbb{F}'}$ such that

$$(4.1.1) \quad M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & au^s \end{pmatrix}$$

with $a \in \mathbb{F}'^\times$ and $0 \leq s < p-1$. We immediately find $\Phi(P_{\text{red}}) = P_{\text{red}}$.

For each $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ we define a (half)-line $\mathcal{L}_z \subset \bar{\mathcal{B}}(m(\mathbf{v}))$ by

$$\begin{aligned} \mathcal{L}_z &= \{[x, m(\mathbf{v})]_z \mid x \geq 0\} \subset \bar{\mathcal{B}}(m(\mathbf{v})) & \text{if } z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}}) \\ \mathcal{L}_\infty &= \{[x, m(\mathbf{v})]_0 \mid x \leq 0\} \subset \bar{\mathcal{B}}(m(\mathbf{v})). \end{aligned}$$

These lines are defined in such a way that

$$(4.1.2) \quad \mathcal{T} := \bigcup_{z \in \mathbb{P}^1(\bar{\mathbb{F}})} \mathcal{L}_z = \bigcup_{z \in \bar{\mathbb{F}}} \mathcal{A}_z \cap \bar{\mathcal{B}}(m(\mathbf{v})).$$

The apartments on the right hand side are given by the basis $e_1, ze_1 + e_2$ and in this basis the semi-linear endomorphism Φ is of the form (4.1.1).

Lemma 4.9. *Let $Q \in \bar{\mathcal{B}}(m(\mathbf{v}))$ be an arbitrary point. Let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(Q, Q') = d_1(Q, \mathcal{T})$. Then*

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (p+1)d_1(Q, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}) \\ d_2(Q, \Phi(Q)) &= (p-1)d_2(Q, P_{\text{red}}). \end{aligned}$$

Proof. The statement on d_2 follows immediately from Lemma 4.7. For the statement on d_1 we assume $Q' \in \mathcal{L}_0$. The cases $Q' \in \mathcal{L}_z$ for $z \in \bar{\mathbb{F}}$ are analogous and the case $Q' \in \mathcal{L}_\infty$ is obtained by interchanging e_1 and e_2 .

First assume $Q = Q'$, i.e. $Q = [x, m(\mathbf{v})]_0 \in \mathcal{L}_0$. Then Lemma 4.7 implies $\Phi(Q) = [px, pm(\mathbf{v}) + 2s]_0$ and hence $d_1(Q, \Phi(Q)) = (p-1)x = (p-1)d_1(Q, P_{\text{red}})$. Now assume $Q \neq Q'$. We write

$$Q = [x, m(\mathbf{v})]_q, \quad Q' = [x', m(\mathbf{v})]_0$$

with $x > x' = v_u(q) \in \mathbb{Z}_{>0}$. Then $\Phi(Q) = [px, pm(\mathbf{v}) + 2s]_{\phi(q)}$ by Lemma 4.7. Using $v_u(\phi(q)) = px'$, we find

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (x - x') + (px' - x') + (px - px') \\ &= (p+1)x - 2x' = (p+1)d_1(Q, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}). \end{aligned}$$

□

Remark 4.10. This Lemma shows that the case of a direct sum of two isomorphic objects corresponds to the case B 2 in [PR2] 6.d:

The unique point fixed by Φ is the point P_{red} and the projection of this point to the building for $PGL_2(\bar{\mathbb{F}}((u)))$ is a vertex. The link of this vertex is the projection of \mathcal{T} and all the half-lines \mathcal{L}_z of \mathcal{T} (for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$) are fixed by Φ .

Proposition 4.11. *With the notations of Definition 4.1 and (4.0.3), (4.0.7) assume that*

$$(M_{\bar{\mathbb{F}}}, \Phi) \cong (\mathfrak{M}^s(a)[\frac{1}{u}], \Phi_a^s) \oplus (\mathfrak{M}^s(a)[\frac{1}{u}], \Phi_a^s)$$

for some $a \in \bar{\mathbb{F}}^\times$ and $0 \leq s < p-1$.

(i) *The schemes $X_{[M']}^\mathbf{v}$ are empty for all $[M'] \in \mathcal{S}(\mathbf{v}) \setminus \{[M_a]\}$.*

(ii) *The scheme $X_{[M_a]}^\mathbf{v}$ is given by*

$$X_{[M_a]}^\mathbf{v} \cong \begin{cases} \emptyset & \text{if } m(\mathbf{v}) + \frac{r_1 - r_2}{p-1} \notin 2\mathbb{Z} \\ \{*\} & \text{if } 0 = \frac{r_1 - r_2}{p-1} \in \mathbb{Z} \text{ and } \frac{r_1 - r_2}{p-1} \equiv m(\mathbf{v}) \pmod{2} \\ \mathbb{P}_{\bar{\mathbb{F}}}^1 & \text{if } 0 \neq \frac{r_1 - r_2}{p-1} \in \mathbb{Z} \text{ and } \frac{r_1 - r_2}{p-1} \equiv m(\mathbf{v}) \pmod{2}. \end{cases}$$

(iii) *If non empty, the scheme $X_0^\mathbf{v}$ is connected.*

Proof. We first claim that every \mathbf{v} -admissible lattice \mathfrak{M} can be linked to a \mathbf{v} -admissible lattice $\mathfrak{M}' \in \mathcal{T}$ by a chain of \mathbb{P}^1 .

Assume $\mathfrak{M} = [x, m(\mathbf{v})]_q \notin \mathcal{T}$ and let $Q' \in \mathcal{T}$ be the unique point satisfying $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{T})$. Without loss of generality, we may again assume that

$Q' = [x', m(\mathbf{v})]_0 \in \mathcal{L}_0$. By construction we have $\mathfrak{M}, Q' \in \mathcal{A}_q$ and we choose the following basis b_1, b_2 of \mathfrak{M} :

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1, \quad b_2 = u^{(m(\mathbf{v})-x)/2} (qe_1 + e_2).$$

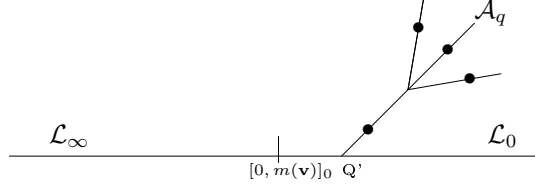


FIGURE 7. The fat points mark the image of the morphism $\bar{\chi}$ in the building in the case $p = 3$ and $\mathbb{F} = \mathbb{F}_3$.

Applying Lemma 3.7 with this basis yields a morphism $\bar{\chi} : \mathbb{P}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$ with $\bar{\chi}(z) = [x, m(\mathbf{v})]_{q+zu^{x-1}}$ for $z \in \bar{\mathbb{F}} = \mathbb{A}^1(\bar{\mathbb{F}})$ and $\bar{\chi}(\infty) = [x-2, m(\mathbf{v})]_q$. We have

$$d_1(\bar{\chi}(\infty), P_{\text{red}}) < d_1(\bar{\chi}(z), P_{\text{red}}) = d_1(\mathfrak{M}, P_{\text{red}})$$

for all $z \in \bar{\mathbb{F}}$, while $d_1(\bar{\chi}(z), T) \leq d_1(\mathfrak{M}, T)$ for all $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ and, by construction, $d_1(\bar{\chi}(\infty), T) < d_1(\mathfrak{M}, T)$.

By Lemma 4.9 and Lemma 2.4, the morphism $\bar{\chi}$ factors through $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ and the claim follows by induction on the distance $d_1(\mathfrak{M}, T)$.

Now we assume that $\mathfrak{M} \in T$ is a \mathbf{v} -admissible lattice and we are looking for a \mathbf{v} -admissible lattice \mathfrak{M}' that can be linked with \mathfrak{M} by a \mathbb{P}^1 and that has strictly smaller distance d_1 from $P_{\text{red}} = [0, \frac{-2s}{p-1}]_0$ than \mathfrak{M} , i.e. $d_1(\mathfrak{M}', P_{\text{red}}) < d_1(\mathfrak{M}, P_{\text{red}})$.

We may assume $\mathfrak{M} = [x, m(\mathbf{v})]_0 \in \mathcal{L}_0$. Assuming $x > 1$, our candidate for \mathfrak{M}' is $[x-2, m(\mathbf{v})]_0$. Fixing a basis

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1, \quad b_2 = u^{(m(\mathbf{v})-x)/2} e_2$$

of \mathfrak{M} so that $\mathfrak{M}' = \langle u^{-1}b_1, ub_2 \rangle$, yields a morphism $\bar{\chi} : \mathbb{P}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$ with $\bar{\chi}(0) = \mathfrak{M}$ and $\bar{\chi}(\infty) = \mathfrak{M}'$. This morphism factors through $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ iff the lattices $\bar{\chi}(z) = [x, m(\mathbf{v})]_{zu^{x-1}}$ are \mathbf{v} -admissible for all $z \in \bar{\mathbb{F}} \setminus \{0\}$. This is the case iff

$$\begin{aligned} d_1(\bar{\chi}(z), \Phi(\bar{\chi}(z))) &= (p+1)d_1(\bar{\chi}(z), P_{\text{red}}) - 2d_1([x-1, m(\mathbf{v})]_0, P_{\text{red}}) \\ &= (p+1)x - 2(x-1) = (p-1)x + 2 \leq r_1 - r_2. \end{aligned}$$

Consider the following subset of \mathbf{v} -admissible lattices

$$\mathcal{N} = \{\mathfrak{M} \in \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}(\bar{\mathbb{F}}) \mid \mathfrak{M} \notin T \text{ or } (\mathfrak{M} \in T \text{ and } d_1(\mathfrak{M}, P_{\text{red}}) \leq \frac{r_1-r_2-2}{p-1})\}.$$

So far, we have shown that all \mathbf{v} -admissible lattices $\mathfrak{M} \in \mathcal{N}$ can either be linked to the lattice $[0, m(\mathbf{v})]_0$ or to one of the lattices

$$(4.1.3) \quad \{[1, m(\mathbf{v})]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-1, m(\mathbf{v})]_0\} = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, P_{\text{red}}) = 1\}.$$

by a chain of \mathbb{P}^1 . Here, the two different cases depend on $m(\mathbf{v}) \bmod 2$. Hence the subset of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$ given by the lattices in \mathcal{N} is connected: using Remark 3.8 again, the set in (4.1.3) forms a \mathbb{P}^1 . The Proposition now follows from the following two facts:

(a) If $\mathfrak{M} \in \mathcal{N}$, then \mathfrak{M} is not \mathbf{v} -ordinary, i.e. $\mathcal{N} \subset X_0^{\mathbf{v}}(\bar{\mathbb{F}})$.

(b) If $\mathfrak{M} \notin \mathcal{N}$ is \mathbf{v} -admissible, then $\mathfrak{M} \in X_{[M_a]}^{\mathbf{v}}(\bar{\mathbb{F}})$ and

$$(4.1.4) \quad \mathfrak{M} \in \{[\frac{r_1-r_2}{p-1}, m(\mathbf{v})]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-\frac{r_1-r_2}{p-1}, m(\mathbf{v})]_0\}.$$

By Remark 3.8, this set forms a \mathbb{P}^1 if $r_1 \neq r_2$. Otherwise it is a single point.

Proof of (a): If $\mathfrak{M} \in \mathcal{T}$, then $d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) \leq r_1 - r_2 - 2 < r_1 - r_2$ and hence the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} are not given by $(e - r_2, e - r_1)$. If $\mathfrak{M} \notin \mathcal{T}$, say $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $x > v_u(q) > 0$ for example, then $\mathfrak{M} = \langle b_1, b_2 \rangle$ with

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1, \quad b_2 = u^{(m(\mathbf{v})-x)/2} (q e_1 + e_2)$$

and one finds

$$\mathfrak{M} \sim (a_{ij})_{ij} = \begin{pmatrix} au^{\frac{p-1}{2}(x+m(\mathbf{v}))+s} & a\phi(q)u^{\frac{p-1}{2}m(\mathbf{v})-\frac{p+1}{2}x+s} \\ 0 & au^{\frac{p-1}{2}(m(\mathbf{v})-x)+s} \end{pmatrix}$$

with $v_u(a_{12}) < v_u(a_{11})$, because $v_u(q) < x$, and hence the minimal elementary divisor of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} is not given by a Φ -stable subspace.

Proof of (b): Let $\mathfrak{M} \notin \mathcal{N}$ be \mathbf{v} -admissible. Then $\mathfrak{M} \in \mathcal{T}$ and

$$\frac{r_1-r_2-2}{p-1} < d_1(\mathfrak{M}, P_{\text{red}}) \leq \frac{r_1-r_2}{p-1}.$$

We show that $d_1(\mathfrak{M}, P_{\text{red}}) = \frac{r_1-r_2}{p-1}$ which implies (4.1.4).

Suppose that $\mathfrak{M} = [x, m(\mathbf{v})]_z$ with $z \in \bar{\mathbb{F}}$ and

$$x = \pm \frac{r_1-r_2-1}{p-1} \in \mathbb{Z}, \quad m(\mathbf{v}) = \frac{2e-d'-2s}{p-1} = \frac{2e-r_1-r_2-2s}{p-1}.$$

In this case we find

$$x + m(\mathbf{v}) = \frac{2e-2s-(r_1+r_2)\pm(r_1-r_2)\mp 1}{p-1} \notin 2\mathbb{Z},$$

contradiction. We are left to show that $\mathfrak{M} \in X_{[M_a]}^{\mathbf{v}}(\bar{\mathbb{F}})$, i.e. that there exists a vector $e_{\mathfrak{M}} \in \mathfrak{M}$ and a Φ -stable subspace $\bar{\mathbb{F}}[u]e_{\mathfrak{M}} \subset \mathfrak{M}$ with $\Phi(e_{\mathfrak{M}}) = au^{e-r_1}e_{\mathfrak{M}}$. An easy computation shows that we may choose

$$\begin{aligned} e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}} (ze_1 + e_2) & \text{if } \mathfrak{M} = [\frac{r_1-r_2}{p-1}, m(\mathbf{v})]_z, \ z \in \bar{\mathbb{F}} \\ e_{\mathfrak{M}} &= u^{\frac{e-r_1-s}{p-1}} e_1 & \text{if } \mathfrak{M} = [-\frac{r_1-r_2}{p-1}, m(\mathbf{v})]_0. \end{aligned}$$

□

We conclude the discussion by determining the cases where $\mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$ is reduced to a single point.

Corollary 4.12. (i) If $m(\mathbf{v}) \equiv 0 \pmod{2}$, then $\mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}} = \{*\}$ iff $\frac{r_1-r_2}{p-1} < 2$.

(ii) If $m(\mathbf{v}) \equiv 1 \pmod{2}$, then $\mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$ can not be a single point.

$$\begin{aligned} \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}} &= \emptyset & \Leftrightarrow 0 \leq \frac{r_1-r_2}{p-1} < 1 \\ \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}} \otimes_{\bar{\mathbb{F}}} \bar{\mathbb{F}} &\cong \mathbb{P}_{\bar{\mathbb{F}}}^1 & \Leftrightarrow 1 \leq \frac{r_1-r_2}{p-1} < 3. \end{aligned}$$

Proof. (i) As $m(\mathbf{v}) \equiv 0 \pmod{2}$, the lattice $[0, m(\mathbf{v})]_0$ is always \mathbf{v} -admissible. It is the unique point of $\mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$ if the lattices \mathfrak{M} with $d_1(\mathfrak{M}, P_{\text{red}}) = 2$ are not \mathbf{v} -admissible. By Lemma 4.9 this is the case iff $\frac{r_1-r_2}{p-1} < 2$.

(ii) The scheme is empty if the lattices \mathfrak{M} with $d_1(\mathfrak{M}, P_{\text{red}}) = 1$ are not \mathbf{v} -admissible. By Lemma 4.9, this is the case iff $\frac{r_1-r_2}{p-1} < 1$.

If $\frac{r_1-r_2}{p-1} \geq 1$, then the lattices

$$\{[1, 0]_z \mid z \in \bar{\mathbb{F}}\} \cup \{[-1, 0]_0\}$$

are \mathbf{v} -admissible and form a $\mathbb{P}_{\mathbb{F}}^1$. Again by Lemma 4.9 there are no other \mathbf{v} -admissible lattices iff $\frac{r_1-r_2}{p-1} < 3$. \square

4.2. The case $(M_{\mathbb{F}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_2, \Phi_2)$. In this section we treat the case where $(M_{\mathbb{F}}, \Phi)$ becomes isomorphic to the direct sum of two non-isomorphic one-dimensional objects after extending the scalars to some finite extension. The situation is the following: There exists a finite extension \mathbb{F}' of \mathbb{F} and a basis e_1, e_2 of $M_{\mathbb{F}'}$ such that

$$M_{\mathbb{F}'} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a, b \in \mathbb{F}'^\times$ and $0 \leq s, t < p-1$. As we are assuming that the direct summands are not isomorphic, we further have $s \neq t$ or $a \neq b$. Again we find $\Phi(P_{\text{red}}) = P_{\text{red}}$.

Lemma 4.13. *Let $Q \in \bar{\mathcal{B}}(m(\mathbf{v}))$ be an arbitrary point. Let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ be the unique point satisfying $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$. Then*

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (p+1)d_1(Q, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}) \\ d_2(Q, \Phi(Q)) &= (p-1)d_2(Q, P_{\text{red}}). \end{aligned}$$

Proof. This is similar to Lemma 4.9. Again the statement on d_2 is an immediate consequence of Lemma 4.7. Let Q be any point. We may assume that the unique point $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$ satisfying $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$ is given by $[x, m(\mathbf{v})]_0$ with $x \geq \frac{t-s}{p-1}$. The case $x \leq \frac{t-s}{p-1}$ is obtained by interchanging e_1 and e_2 .

If $Q = Q'$, then $Q \in \mathcal{A}_0$ and the statement is a consequence of Lemma 4.7.

Assume $Q \neq Q'$ and put

$$Q = [x, m(\mathbf{v})]_q, \quad Q' = [x', m(\mathbf{v})]_0$$

with $x > x' = v_u(q) \geq \frac{t-s}{p-1}$. Now $\Phi(Q) = [px + s - t, pm(\mathbf{v}) + s + t]_{q'}$ with $q' = ab^{-1}u^{-(t-s)}\phi(q)$.

If $s \neq t$, then $x' = v_u(q) > \frac{t-s}{p-1}$ or equivalently $x' = v_u(q) < v_u(q') = pv_u(q) - (t-s)$, and we find

$$\begin{aligned} d_1(Q, \Phi(Q)) &= (x - x') + (px' - (t-s) - x') + (px + s - t - (px' + s - t)) \\ &= (p+1)(x - \frac{t-s}{p-1}) - 2(x' - \frac{t-s}{p-1}) \\ &= (p+1)d_1(Q, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}). \end{aligned}$$

If $s = t$, then $a \neq b$. We find $v_u(q) \neq v_u(q')$ if $v_u(q) \neq 0$ and in this case the computation is the same as above.

If $q = a_0 + a_1u + \dots$ with $a_0 \neq 0$, then $q' = ab^{-1}a_0 + \dots$ and hence the absolute coefficient of q is different from the absolute coefficient of q' . The geodesic between Q and the projection of $\Phi(Q)$ to $\bar{\mathcal{B}}(m(\mathbf{v}))$ contains the point $Q' = [0, m(\mathbf{v})]_0 = [\frac{t-s}{p-1}, m(\mathbf{v})]_0$. Hence

$$\begin{aligned} d_1(Q, \Phi(Q)) &= x + (px + s - t) = (p+1)x \\ &= (p+1)d_1(Q, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}). \end{aligned}$$

\square

Remark 4.14. Again, this Lemma shows the connection to [PR2] 6.d. The point fixed by Φ is again the point P_{red} .

If $s = t$, then we are in the case B 2 of loc. cit.: The projection of the fixed point to the building for $PGL_2(\mathbb{F}((u)))$ is a vertex. Exactly two of the half-lines of the

link of this vertex are fixed by Φ .

If $s \neq t$ we are in the case A 2 of loc. cit.: The projection of the fixed point P_{red} is not a vertex but it lies on an edge and the projections of the two half-lines $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{t-s}{p-1}\}$ and $\{[x, m(\mathbf{v})]_0 \mid x \geq \frac{t-s}{p-1}\}$ to the building for $PGL_2(\bar{\mathbb{F}}((u)))$ are fixed by Φ .

Proposition 4.15. *With the notations of Definition 4.1 and (4.0.3), (4.0.7) assume that*

$$(M_{\bar{\mathbb{F}}}, \Phi) \cong (\mathfrak{M}^s(a)[\frac{1}{u}], \Phi_a^s) \oplus (\mathfrak{M}^t(b)[\frac{1}{u}], \Phi_b^t)$$

with $a, b \in \bar{\mathbb{F}}^\times$ and $0 \leq s, t < p-1$. Further assume $a \neq b$ or $s \neq t$.

(i) The schemes $X_{[M']}^\mathbf{v}$ are empty for all $[M'] \in \mathcal{S}(\mathbf{v}) \setminus \{[M_a], [M_b]\}$.

(ii) If $s = t$, then

$$X_{[M_a]}^\mathbf{v} \cong X_{[M_b]}^\mathbf{v} = \begin{cases} \emptyset & \text{if } m(\mathbf{v}) + \frac{r_1-r_2}{p-1} \notin 2\mathbb{Z} \\ \{*\} & \text{if } m(\mathbf{v}) + \frac{r_1-r_2}{p-1} \in 2\mathbb{Z}, \end{cases}$$

further $X_{[M_a]}^\mathbf{v} = X_{[M_b]}^\mathbf{v}$ iff $r_1 = r_2$.

(iii) If $s \neq t$, then

$$X_{[M_a]}^\mathbf{v} = \begin{cases} \emptyset & \text{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \text{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z} \end{cases}$$

$$X_{[M_b]}^\mathbf{v} = \begin{cases} \emptyset & \text{if } \frac{t-s}{p-1} + \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \text{if } \frac{t-s}{p-1} + \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}. \end{cases}$$

(iv) If non empty the scheme $X_0^\mathbf{v}$ is connected.

Proof. Again, this is similar to the proof of Proposition 4.11.

First, we link any \mathbf{v} -admissible lattice to a \mathbf{v} -admissible lattice in \mathcal{A}_0 by a chain of \mathbb{P}^1 .

Let \mathfrak{M} be any \mathbf{v} -admissible lattice and let $Q' \in \mathcal{A}_0 \cap \bar{B}(m(\mathbf{v}))$ be the unique point with $d_1(Q, Q') = d_1(Q, \mathcal{A}_0)$. Again, we may assume without loss of generality $Q' = [x', m(\mathbf{v})]_0$ with $x' \geq \frac{t-s}{p-1}$. Completely analogous to the proof of Proposition 4.11, we find a morphism

$$\bar{\chi} : \mathbb{P}_{\bar{\mathbb{F}}}^1 \rightarrow \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$$

such that $\bar{\chi}(0) = \mathfrak{M}$ and $d_1(\bar{\chi}(\infty), \mathcal{A}_0) < d_1(\mathfrak{M}, \mathcal{A}_0)$. By induction on the distance $d_1(\mathfrak{M}, \mathcal{A}_0)$, we find that we can link any \mathbf{v} -admissible lattice to a \mathbf{v} -admissible lattice in \mathcal{A}_0 .

Now assume $\mathfrak{M} = [x, m(\mathbf{v})]_0 \in \mathcal{A}_0$.

If $x > \frac{t-s}{p-1}$, we find a map $\mathbb{P}^1 \rightarrow \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$, as in the proof of Proposition 4.11, whose image contains $[x, m(\mathbf{v})]_0$ and $[x-2, m(\mathbf{v})]_0$, if $(p-1)(x - \frac{t-s}{p-1}) \leq r_1 - r_2 - 2$.

If $x < \frac{t-s}{p-1}$, we find a map $\mathbb{P}^1 \rightarrow \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}$ whose image contains $[x, m(\mathbf{v})]_0$ and $[x+2, m(\mathbf{v})]_0$, if $(p-1)(\frac{t-s}{p-1} - x) \leq r_1 - r_2 - 2$.

Similarly as in Proposition 4.11, one can proof the following two facts:

(a) The set

$$\mathcal{N} = \{\mathfrak{M} \in \mathcal{GR}_{V_{\bar{\mathbb{F}},0}}^{\mathbf{v},\text{loc}}(\bar{\mathbb{F}}) \mid \mathfrak{M} \notin \mathcal{A}_0 \text{ or } (\mathfrak{M} \in \mathcal{A}_0 \text{ and } d_1(\mathfrak{M}, P_{\text{red}}) \leq \frac{r_1-r_2-2}{p-1})\}$$

consists of non- \mathbf{v} -ordinary lattices.

(b) If $\mathfrak{M} \in \mathcal{A}_0$ is \mathbf{v} -admissible, then

$$d_1(\mathfrak{M}, P_{\text{red}}) > \frac{r_1-r_2-2}{p-1} \Rightarrow d_1(\mathfrak{M}, P_{\text{red}}) = \frac{r_1-r_2}{p-1}.$$

Now \mathcal{N} defines a connected subset of $X_0^{\mathbf{v}}$:

If $s \neq t$, then there is a unique lattice with minimal distance from $[\frac{t-s}{p-1}, m(\mathbf{v})]_0$ and every \mathbf{v} -admissible lattice in \mathcal{N} can be linked to this lattice by a chain of \mathbb{P}^1 .

If $s = t$, then either $[\frac{t-s}{p-1}, m(\mathbf{v})]_0$ is a lattice itself and any \mathbf{v} -admissible lattice can be linked to this lattice by a chain of \mathbb{P}^1 , or there are two \mathbf{v} -admissible lattices $[\pm 1, m(\mathbf{v})]_0$ in \mathcal{N} with distance 1 from $[0, m(\mathbf{v})]_0$ and by the above there is a morphism

$$\mathbb{P}^1 \rightarrow \mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$$

containing both lattices in its image. Thus \mathcal{N} defines a connected subset. Consider the following points:

$$\begin{aligned} Q_+ &= [\frac{t-s}{p-1} + \frac{r_1-r_2}{p-1}, m(\mathbf{v})]_0 \\ Q_- &= [\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1}, m(\mathbf{v})]_0. \end{aligned}$$

We are left to show that, if one of these points defines a lattice $\mathfrak{M}_+ = Q_+$ (resp. $\mathfrak{M}_- = Q_-$), then this point lies in $X_{[M_b]}^{\mathbf{v}}$ (resp. $X_{[M_a]}^{\mathbf{v}}$).

If $\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then $Q_- = \mathfrak{M}_-$ is a lattice and

$$e_{\mathfrak{M}_-} = u^{(e-r_1-s)/(p-1)} e_1$$

defines a Φ -stable subspace satisfying $\Phi(e_{\mathfrak{M}_-}) = au^{e-r_1}e_{\mathfrak{M}_-}$, i.e. $\mathfrak{M}_- \in X_{[M_a]}^{\mathbf{v}}$.

If $\frac{t-s}{p-1} + \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then $Q_+ = \mathfrak{M}_+$ is a lattice and

$$e_{\mathfrak{M}_+} = u^{(e-r_1-t)/(p-1)} e_2$$

defines a Φ -stable subspace satisfying $\Phi(e_{\mathfrak{M}_+}) = bu^{e-r_1}e_{\mathfrak{M}_+}$, i.e. $\mathfrak{M}_+ \in X_{[M_b]}^{\mathbf{v}}$.

We have two different cases:

If $s = t$ and $\frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$, then the lattices \mathfrak{M}_+ and \mathfrak{M}_- define points $\mathfrak{M}_- \in X_{[M_a]}^{\mathbf{v}}$ and $\mathfrak{M}_+ \in X_{[M_b]}^{\mathbf{v}}$ which coincide iff $r_1 = r_2$.

If $s \neq t$, then

$$Q_- \text{ defines an isolated point in } X_{[M_a]}^{\mathbf{v}} \Leftrightarrow \frac{t-s-(r_1-r_2)}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$$

$$Q_+ \text{ defines an isolated point in } X_{[M_b]}^{\mathbf{v}} \Leftrightarrow \frac{t-s+(r_1-r_2)}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}.$$

This cannot happen at the same time, as $\frac{t-s}{p-1} \notin \mathbb{Z}$. This finishes the proof of the Proposition. \square

Corollary 4.16. (i) Assume $s = t$.

(a) If $m(\mathbf{v}) \equiv 0 \pmod{2}$, then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \{*\}$ iff $\frac{r_1-r_2}{p-1} < 2$.

(b) If $m(\mathbf{v}) \equiv 1 \pmod{2}$, then $\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}}$ cannot be a single point.

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \emptyset \Leftrightarrow 0 \leq \frac{r_1-r_2}{p-1} < 1$$

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \{*\} \cup \{*\} \Leftrightarrow 1 \leq \frac{r_1-r_2}{p-1} < 3.$$

(ii) Assume $s \neq t$.

Define $x_0 = \lfloor \frac{t-s}{p-1} \rfloor$ and write $\xi = \frac{t-s}{p-1} - x_0$ for the fractional part of $\frac{t-s}{p-1}$.

$$\mathcal{GR}_{V_{\mathbb{F}}, 0}^{\mathbf{v}, \text{loc}} = \{*\} \Leftrightarrow \begin{cases} 0 + \xi \leq \frac{r_1-r_2}{p-1} < 2 - \xi & \text{if } m(\mathbf{v}) \equiv x_0 \pmod{2} \\ 1 - \xi \leq \frac{r_1-r_2}{p-1} < 1 + \xi & \text{if } m(\mathbf{v}) \not\equiv x_0 \pmod{2}. \end{cases}$$

Proof. (i) This is nearly identical to Corollary 4.12.

(ii) Assume $m(\mathbf{v}) \equiv x_0 \pmod{2}$. Then $[x_0, m(\mathbf{v})]_0$ is the unique lattice with minimal distance d_1 from P_{red} . By Lemma 4.13 it is \mathbf{v} -admissible iff $\frac{r_1-r_2}{p-1} \geq \xi$.

Again by Lemma 4.13 it is the only \mathbf{v} -admissible lattice iff $[x_0 + 2, m(\mathbf{v})]_0$ is not \mathbf{v} -admissible. This is the case iff $\frac{r_1-r_2}{p-1} < 2 - \xi$.

The case $m(\mathbf{v}) \not\equiv x_0 \pmod{2}$ is similar. \square

4.3. The case of a non split extension. Finally, we analyze the case where $(M_{\bar{\mathbb{F}}}, \Phi)$ is a non split extension of two one dimensional objects. There is a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

with $0 \leq s, t < p-1$ and $a, b \in \bar{\mathbb{F}}^\times$, $\gamma \in \bar{\mathbb{F}}((u))$. In any basis of the form $e_1, qe_1 + e_2$ defining the apartment \mathcal{A}_q , the endomorphism Φ is upper triangular with diagonal entries au^s and bu^t , and we fix the basis such that the valuation of the upper right entry $k := v_u(\gamma)$ is maximal.

Lemma 4.17. (i) *The integer $k = v_u(\gamma)$ satisfies*

$$k \leq \frac{pt-s}{p-1}.$$

(ii) *If $\mathfrak{M} = [x, y]_q$ with $\min\{x, v_u(q)\} \geq \frac{k-s}{p}$, then*

$$d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (p+1)x + s + t - 2k$$

$$d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (p-1)d_2(\mathfrak{M}, P_{\text{red}}).$$

(iii) *If $\mathfrak{M} = [x, y]_q$ with $x < \frac{k-s}{p}$ or $v_u(q) < \frac{k-s}{p}$, let $Q' \in \mathcal{A}_0 \cap \bar{\mathcal{B}}(y)$ be the unique point such that $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{A}_0)$. Then*

$$d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (p+1)d_1(\mathfrak{M}, P_{\text{red}}) - 2d_1(Q', P_{\text{red}})$$

$$d_2(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (p-1)d_2(\mathfrak{M}, P_{\text{red}}).$$

Proof. (i) This follows from the maximality of $k = v_u(\gamma)$: We have

$$(4.3.1) \quad \Phi(qe_1 + e_2) = (\gamma + au^s\phi(q) - bu^tq)e_1 + bu^t(qe_1 + e_2).$$

And

$$v_u(au^s\phi(q) - bu^tq) = \begin{cases} v_u(q) + t & \text{if } v_u(q) > \frac{t-s}{p-1} \\ pv_u(q) + s & \text{if } v_u(q) < \frac{t-s}{p-1}. \end{cases}$$

If we had $k = v_u(\gamma) = v_u(q) + t$ for any q with $v_u(q) > \frac{t-s}{p-1}$, we could delete the leading coefficient of γ in (4.3.1) which contradicts the maximality of $v_u(\gamma)$. Hence we have $k < v_u(q) + t$ for all q with $v_u(q) > \frac{t-s}{p-1}$ which yields the first claim.

(ii) The first part of the lemma implies $\frac{k-s}{p} \geq k-t$ and hence our assumptions on $v_u(q)$ imply $k \leq \min\{v_u(q) + t, pv_u(q) + s\}$. We find $v_u(\gamma + au^s\phi(q) - bu^tq) = v_u(\gamma)$ and we may assume $q = 0$, i.e. $\mathfrak{M} \in \mathcal{A}_0$, as the situation is the same as in the standard apartment. Now we have $\langle \Phi(\mathfrak{M}) \rangle = [px + s - t, py + s + t]_{b^{-1}u^{-t}\gamma}$, and $x \geq \frac{k-s}{p}$ implies

$$px + s - t \geq k - t, \quad x \geq k - t.$$

Thus $d_1(\mathfrak{M}, \langle \Phi(\mathfrak{M}) \rangle) = (px + s - t - (k - t)) + (x - (k - t)) = (p + 1)x + s + t - 2k$. The statement on d_2 is easy.

(iii) If $\mathfrak{M} \notin \mathcal{A}_0$, then $v_u(q) < \frac{k-s}{p} \leq \frac{t-s}{p-1}$ and hence

$$v_u(\gamma + au^s\phi(q) - bu^tq) = v_u(au^s\phi(q) - bu^tq)$$

and the situation is the same as in the split case, i.e. the case $\gamma = 0$. If $\mathfrak{M} \in \mathcal{A}_0$, then $\langle \Phi(\mathfrak{M}) \rangle \in \mathcal{A}_0$ and the statement is easy. \square

Remark 4.18. In the case of a non split extension we are in the case B 2 or A 3 of [PR2] 6.d. More precisely, if $\frac{k-s}{p} \notin \mathbb{Z}$, then the unique fixed point of ([PR2], Prop. 6.1) is not in the building $\tilde{\mathcal{B}}$. It is only visible after extending $\tilde{\mathbb{F}}((u))$ to some separable wildly ramified extension (the apartment containing the fixed point will branch off from \mathcal{A}_0 at the line $x = \frac{k-s}{p}$, because we can successively delete the leading coefficient of γ in (4.3.1) if there is some q with $v_u(q) = \frac{k-s}{p}$). The image of the half line $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{k-s}{p}\}$ in the building for $PGL_2(\tilde{\mathbb{F}}((u)))$ is stable under Φ and the geodesic between $[\lfloor \frac{k-s}{p} \rfloor + 1, m(\mathbf{v})]_0$ and its image under Φ contains the (projection of the) point $[x_0, m(\mathbf{v})]_0$ in the building for $PGL_2(\tilde{\mathbb{F}}((u)))$. This is the case A 3 of [PR2] 6.d.

If $\frac{k-s}{p} \in \mathbb{Z}$, then we are in the case B 2 of [PR2] 6.d.: In this case the maximality of $k = v_u(q)$ implies $k - t = \frac{k-s}{p} = \frac{t-s}{p-1} = 0$ (otherwise we could delete the leading coefficient of γ) and we find that P_{red} is the fixed point in the building. In this case there is a unique half-line in the apartment for $PGL_2(\tilde{\mathbb{F}}((u)))$ that is fixed by Φ , namely the image of the half-line $\{[x, m(\mathbf{v})]_0 \mid x \leq 0\}$ under the projection.

Proposition 4.19. *With the notations of Definition 4.1 and (4.0.3), (4.0.7), assume that $(M_{\tilde{\mathbb{F}}}, \Phi)$ is a non split extension*

$$0 \rightarrow (\mathfrak{M}^s(a)[\frac{1}{u}], \Phi_a^s) \rightarrow (M_{\tilde{\mathbb{F}}}, \Phi) \rightarrow (\mathfrak{M}^t(b)[\frac{1}{u}], \Phi_b^t) \rightarrow 0$$

for some $a, b \in \tilde{\mathbb{F}}^\times$ and $0 \leq s, t < p - 1$.

(i) *The schemes $X_{[M']}^\mathbf{v}$ are empty for all $[M'] \in \mathcal{S}(\mathbf{v}) \setminus \{[M_a]\}$.*

(ii) *For $X_{[M_a]}^\mathbf{v}$ the following holds:*

$$X_{[M_a]}^\mathbf{v} = \begin{cases} \emptyset & \text{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \notin 2\mathbb{Z} \\ \{*\} & \text{if } \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}. \end{cases}$$

(iii) *If non empty, the scheme $X_0^\mathbf{v}$ is connected.*

Proof. Lemma 4.17 (i) implies $\frac{k-s}{p} \leq \frac{t-s}{p-1}$ and an easy computation using the same inequality shows that

$$\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} \leq \frac{k-s}{p} \Leftrightarrow \frac{k-s}{p} \leq \frac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

and hence

$$\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} \leq \frac{k-s}{p} \leq \frac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

if $\mathcal{GR}_{V_{\tilde{\mathbb{F}},0}}^{\mathbf{v},\text{loc}} \neq \emptyset$. Further denote by $\tilde{\mathcal{N}}$ the set of \mathbf{v} -admissible lattices $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $\frac{k-s}{p} \leq \min\{x, v_u(q)\}$.

As we have seen above, the situation for the \mathbf{v} -admissible lattices $\mathfrak{M} \notin \tilde{\mathcal{N}}$ is the same as in the split case. Hence we can link all \mathbf{v} -admissible lattices to \mathbf{v} -admissible lattices in \mathcal{A}_0 by a chain of \mathbb{P}^1 . If $\mathfrak{M} = [x, m(\mathbf{v})]_0$ is a \mathbf{v} -admissible lattice in \mathcal{A}_0

with $x + 2 < \frac{k-s}{p}$, then there is a \mathbb{P}^1 in $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ containing $\mathfrak{M} = [x, m(\mathbf{v})]_0$ and $[x + 2, m(\mathbf{v})]_0$, except if $\mathfrak{M} = \mathfrak{M}_- = [\frac{t-s-(r_1-r_2)}{p-1}, m(\mathbf{v})]_0$ which defines an isolated point in $X_{[M_a]}^{\mathbf{v}}$ if $\frac{t-s}{p-1} - \frac{r_1-r_2}{p-1} + m(\mathbf{v}) \in 2\mathbb{Z}$ (compare Proposition 4.15).

Let $\mathfrak{M}' = [x_0, m(\mathbf{v})]_0$ be the lattice where x_0 is the maximal integer smaller than $\frac{k-s}{p}$ that is congruent to $m(\mathbf{v}) \pmod{2}$. We claim:

(a) If \mathfrak{M}' is \mathbf{v} -admissible and $x_0 \neq \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1}$, then any lattice in $\tilde{\mathcal{N}}$ can be linked to \mathfrak{M}' by a chain of \mathbb{P}^1 .

(b) The lattices $\mathfrak{M} \in \tilde{\mathcal{N}}$ are non- \mathbf{v} -ordinary.

This finishes the proof of the proposition.

Proof of (a): Let $\mathfrak{M} = [x, m(\mathbf{v})]_q \in \tilde{\mathcal{N}}$ be a lattice. Without loss of generality, we may assume $\mathfrak{M} \in \mathcal{A}_0$, as the situation is the same in all apartments \mathcal{A}_q with $v_u(q) \geq \frac{k-s}{p}$. By Lemma 4.17, we have

$$\frac{k-s}{p} \leq x \leq \frac{1}{p+1}(r_1 - r_2 - s - t + 2k).$$

We consider the basis

$$b_1 = u^{(x+m(\mathbf{v}))/2} e_1, \quad b_2 = u^{(m(\mathbf{v})-x)/2} e_2$$

of \mathfrak{M} and by Lemma 3.7, there is a morphism

$$\bar{\chi} : \mathbb{P}_{\mathbb{F}}^1 \rightarrow \text{Grass } M_{\mathbb{F}}$$

with $\bar{\chi}(z) = [x, m(\mathbf{v})]_{z^{x-1}}$ for $z \in \mathbb{F}$ and $\bar{\chi}(\infty) = [x - 2, m(\mathbf{v})]_0$.

If $x - 1 \geq \frac{k-s}{p}$, then the morphism factors over $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$. Consider the following two cases:

If $\frac{k-s}{p} \leq x_0 + 1$, then this argument shows that we can link all $\mathfrak{M} \in \tilde{\mathcal{N}}$ to the lattice $[x_0, m(\mathbf{v})]_0$ by a chain of \mathbb{P}^1 .

If $\frac{k-s}{p} > x_0 + 1$, then this argument shows that we can link all $\mathfrak{M} \in \tilde{\mathcal{N}}$ to the lattice $\mathfrak{M}'' = [x_0 + 2, m(\mathbf{v})]_0$ by a chain of \mathbb{P}^1 . We can link the lattice \mathfrak{M}'' to the lattice $\mathfrak{M}' = [x_0, m(\mathbf{v})]_0$ if the lattices $\mathfrak{M}_z = [x_0, m(\mathbf{v})]_{zu^{x-1}}$ are \mathbf{v} -admissible for all $z \in \mathbb{F}$. For $z \neq 0$ we have

$$d_1(\mathfrak{M}_z, \langle \Phi(\mathfrak{M}_z) \rangle) = (p+1)d_1(\mathfrak{M}_z, P_{\text{red}}) - 2d_1(Q', P_{\text{red}}),$$

where $Q' = [x_0 + 1, m(\mathbf{v})]_0$ is the unique point in \mathcal{A}_0 with minimal distance from \mathfrak{M}_z . Hence $d_1(\mathfrak{M}_z, \langle \Phi(\mathfrak{M}_z) \rangle) = d_1(\mathfrak{M}', \langle \Phi(\mathfrak{M}') \rangle) + 2$ and the morphism factors through $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ as $x_0 \neq \frac{t-s}{p-1} - \frac{r_1-r_2}{p-1}$. (Otherwise \mathfrak{M}' is the unique isolated point in $X_{[M_a]}^{\mathbf{v}}$).

Proof of (b): Let $\mathfrak{M} \in \tilde{\mathcal{N}}$ be a lattice. Similarly to the proof of Proposition 4.11, we find

$$\mathfrak{M} \sim \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$$

with $v_u(a_{12}) < v_u(a_{11})$ and hence the minimal elementary divisor of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} is not defined by a Φ -stable subspace. \square

Summarizing the results on the connected components we find the following Theorem.

Theorem 4.20. *Assume that $(M_{\mathbb{F}}, \Phi)$ becomes reducible after extending the scalars to some finite extension \mathbb{F}' of \mathbb{F} .*

(i) *The subschemes $X_0^{\mathbf{v}}$ and $X_{[M']}^{\mathbf{v}}$ are open and closed in $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \mathbb{F}'$ for all*

isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$.

(ii) If non empty, the scheme $X_0^\mathbf{v}$ is connected.

(iii) For each $[M'] \in \mathcal{S}(\mathbf{v})$ the scheme $X_{[M']}^\mathbf{v}$ is either empty, a single point or isomorphic to $\mathbb{P}_{\mathbb{F}}^1$.

(iv) There are at most two isomorphism classes $[M'] \in \mathcal{S}(\mathbf{v})$ such that $X_{[M']}^\mathbf{v} \neq \emptyset$.

Proof. This is a summary of the Propositions 4.11, 4.15 and 4.19. \square

This theorem implies a modified version of the conjecture of Kisin stated in ([Ki], 2.4. 16).

Definition 4.21. For an integer s denote by $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},s}$ the open and closed subscheme of $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}}$ consisting of all \mathbf{v} -admissible lattices \mathfrak{M} , where the rank of the maximal Φ -stable subobject \mathfrak{M}_1 satisfying $\langle \Phi(\mathfrak{M}_1) \rangle = u^{e-r_1}\mathfrak{M}_1$ is equal to s .

Corollary 4.22. Assume $p \neq 2$ and let $\rho : G_K \rightarrow V_{\mathbb{F}}$ be any two-dimensional continuous representation of G_K that admits a finite flat model after possibly extending the scalars to some finite extension of \mathbb{F} .

Assume that $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'})$ is a simple algebra for all finite extensions \mathbb{F}' of \mathbb{F} . Then $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},s}$ is geometrically connected for all s . Furthermore

(i) If $s = 1$ and $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ for all finite extensions \mathbb{F}' of \mathbb{F} , then $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},s}$ is either empty or a single point.

If $s = 1$ and $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ for some finite extension \mathbb{F}' of \mathbb{F} , then $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},s}$ is either empty or becomes isomorphic to $\mathbb{P}_{\mathbb{F}'}^1$ after extending the scalars to \mathbb{F}' .

(ii) If $s = 2$, then $\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},s}$ is either empty or a single point.

Proof. Our definitions imply

$$\mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},0} \otimes_{\mathbb{F}} \bar{\mathbb{F}} = X_0^\mathbf{v}.$$

Further

$$\bigcup_{[M'] \in \mathcal{S}(\mathbf{v})} X_{[M']}^\mathbf{v} = \begin{cases} \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},1} & \text{if } r_1 > r_2 \\ \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc},2} & \text{if } r_1 = r_2. \end{cases}$$

By ([Br], Thm. 3.4.3) we have $\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \text{End}_{\mathbb{F}'((u)),\Phi}(M_{\mathbb{F}'})$. The same Theorem implies that the image of the category of finite flat G_K -representations on finite length \mathbb{Z}_p -algebras under the restriction to G_{K_∞} is closed under subobjects and quotients. Hence $V_{\mathbb{F}'}$ is irreducible (resp. reducible, resp. split reducible) if and only if $(M_{\mathbb{F}'}, \Phi)$ is. An easy computation yields:

$\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}'$ if $V_{\mathbb{F}'}$ is irreducible or non-split reducible.

$\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = \mathbb{F}' \times \mathbb{F}'$ if $V_{\mathbb{F}'}$ is the direct sum of two non-isomorphic one-dimensional representations.

$\text{End}_{\mathbb{F}'[G_K]}(V_{\mathbb{F}'}) = M_2(\mathbb{F}')$ if $V_{\mathbb{F}'}$ is the direct sum of two isomorphic one-dimensional representations.

The Corollary now follows from Theorem 4.20 and Propositions 4.11, 4.15 and 4.19. \square

5. THE STRUCTURE OF $X_0^{\mathbf{v}}$

In this section we want to analyze the structure of the connected component $X_0^{\mathbf{v}}$ of non- \mathbf{v} -ordinary lattices. In the absolutely simple case we have

$$X_0^{\mathbf{v}} = \mathcal{GR}_{V_{\mathbb{F}},0}^{\mathbf{v},\text{loc}} \otimes_{\mathbb{F}} \bar{\mathbb{F}}$$

and this is isomorphic to a Schubert variety. In the reducible case it turns out that this component has a quite complicated structure. It is in general not irreducible and its irreducible components have varying dimensions.

5.1. The case $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_1, \Phi_1)$. We assume that $(M_{\bar{\mathbb{F}}}, \Phi)$ is a direct summand of two isomorphic one-dimensional objects and we will use the notations of section 4.1. First we define some subsets of the affine Grassmannian.

Denote by n the maximal integer congruent to $m(\mathbf{v}) \pmod 2$, such that

$$(5.1.1) \quad n \leq \frac{r_1 - r_2 + 2}{p+1}.$$

Denote by l the minimal integer such that

$$(5.1.2) \quad n + 2 \leq \frac{r_1 - r_2 + 2l}{p+1}.$$

For $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ and $j \geq 0$ we define the following points:

$$\begin{aligned} Q_j^z &= [l + (p+1)j, m(\mathbf{v})]_z \quad \text{if } z \in \bar{\mathbb{F}} \\ Q_j^\infty &= [-l - (p+1)j, m(\mathbf{v})]_0. \end{aligned}$$

We define the following subschemes $Z, Z_j \subset X_0^{\mathbf{v}}$ for $j \geq 0$ by specifying its closed points:

$$(5.1.3) \quad \begin{aligned} Z(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, P_{\text{red}}) \leq n\} \\ Z_j(\bar{\mathbb{F}}) &= \bigcup_{z \in \mathbb{P}^1(\bar{\mathbb{F}})} \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^z) \leq n + 2 - l - (p-1)j\}. \end{aligned}$$

We want to consider these subsets as subschemes with the reduced scheme structure.

Lemma 5.1. *With the notations of (5.1.1)-(5.1.3):*

$$X_0^{\mathbf{v}}(\bar{\mathbb{F}}) = \left(\bigcup_{j \geq 0} Z_j(\bar{\mathbb{F}}) \right) \cup Z(\bar{\mathbb{F}}).$$

Proof. Let $\mathfrak{M} = [x, m(\mathbf{v})]_q$ be a non- \mathbf{v} -ordinary lattice and denote by $Q' = [x', m(\mathbf{v})]_z$ the unique lattice with $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{T})$. Without loss of generality, we may assume that $Q' \in \mathcal{L}_0$, i.e. $z = 0$ and $v_u(q) = x' > 0$.

If $1 \leq x' = v_u(q) < l$, then by Lemma 4.9 and the definition of n and l we find that \mathfrak{M} is \mathbf{v} -admissible (and non- \mathbf{v} -ordinary) iff $d_1(\mathfrak{M}, P_{\text{red}}) \leq n$.

If $v_u(q) \geq l$, then there is a unique j such that $l + (p+1)j \leq x' < l + (p+1)(j+1)$. By Lemma 4.9, we find that \mathfrak{M} is \mathbf{v} -admissible iff $d_1(\mathfrak{M}, P_{\text{red}}) \leq \frac{r_1 - r_2 + 2x'}{p+1}$. Now

$$d_1(\mathfrak{M}, P_{\text{red}}) = d_1(\mathfrak{M}, Q_j^0) + (l + (p+1)j)$$

and hence \mathfrak{M} is \mathbf{v} -admissible iff

$$d_1(\mathfrak{M}, Q_j^0) \leq \frac{r_1 - r_2 + 2l}{p+1} + \frac{2(x' - l - (p+1)j)}{p+1} - l - (p-1)j.$$

By the definition of n and l and the fact $x' - l - (p+1)j < (p+1)$ we find that \mathfrak{M} is \mathbf{v} -admissible iff

$$d_1(\mathfrak{M}, Q_j^0) \leq n + 2 - l - (p-1)j.$$

This yields the claim. \square

Proposition 5.2. *With the notation of (5.1.1)-(5.1.3):*

$$(5.1.4) \quad X_0^\vee = \left(\bigcup_{j \geq 0} Z_j \right) \cup Z.$$

- (i) *The scheme Z is isomorphic to an n -dimensional Schubert variety.*
- (ii) *For $j \geq 0$ there is a projective, surjective and birational morphism*

$$f_j : \mathbb{P}_{\mathbb{F}}^1 \times Y_j \rightarrow Z_j$$

where Y_j is an $n + 2 - l - (p - 1)j$ dimensional Schubert variety. Especially Z_j is closed and irreducible.

- (iii) *If $l \neq 2$, then (5.1.4) is the decomposition of X_0^\vee into its irreducible components.*
- (iv) *If $l = 2$, then the decomposition of X_0^\vee into its irreducible components is given by*

$$X_0^\vee = \bigcup_{j \geq 0} Z_j.$$

- (v) *The dimension of X_0^\vee is given by*

$$\dim X_0^\vee = \begin{cases} n + 1 & \text{if } l = 2 \\ n & \text{if } l \neq 2. \end{cases}$$

Proof. (i) The closed points of the scheme Z are the lattices with distance smaller than n from the point $[0, m(\mathbf{v})]_0$. By the same argument as in the proof of Theorem 3.9 (b), this is an n -dimensional Schubert variety.

(ii) The scheme Z_j is the union of the Schubert varieties consisting of the lattices \mathfrak{M} with distance $d_1(\mathfrak{M}, Q_j^z) \leq n + 2 - l - (p - 1)j =: n_j$ for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$.

Let us first assume that $m(\mathbf{v}) \equiv x_j := l + (p + 1)j \pmod{2}$, i.e. Q_j^z is a lattice for all $z \in \mathbb{P}^1(\bar{\mathbb{F}})$. For any linearly independent vectors b_1 and b_2 denote by

$$\psi(b_1, b_2) : Y_j \hookrightarrow \text{Grass } M_{\bar{\mathbb{F}}}$$

the inclusion of the Schubert variety of lattices \mathfrak{M} with

$$\begin{aligned} d_1(\mathfrak{M}, \langle b_1, b_2 \rangle) &\leq n_j \\ d_2(\mathfrak{M}, \langle b_1, b_2 \rangle) &= 0. \end{aligned}$$

First we construct a morphism

$$\tilde{f}_j : \mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j \rightarrow \text{Grass } M_{\bar{\mathbb{F}}}.$$

The inclusion $\psi(e_1, e_2)$ defines a sheaf \mathcal{M}_{Y_j} of $\mathcal{O}_{Y_j}[[u]]$ -lattices in $M_{\bar{\mathbb{F}}} \hat{\otimes}_{\bar{\mathbb{F}}} \mathcal{O}_{Y_j}$. If $U = \text{Spec } A \subset Y_j$ is an affine open we write $\mathfrak{M}_A = \Gamma(U, \mathcal{M}_{Y_j})$ for the $A[[u]]$ -lattice in $M_{\bar{\mathbb{F}}} \hat{\otimes}_{\bar{\mathbb{F}}} A$ defined by \mathcal{M}_{Y_j} . To define the morphism \tilde{f}_j we define a sheaf $\tilde{\mathcal{M}}$ of $\mathcal{O}_{\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j}[[u]]$ -lattices in $M_{\bar{\mathbb{F}}} \hat{\otimes}_{\bar{\mathbb{F}}} \mathcal{O}_{\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j}$. Let $\mathbb{P}_{\bar{\mathbb{F}}}^1 = V_0 \cup V_\infty$ with $V_0 = \text{Spec } \bar{\mathbb{F}}[T]$ and $V_\infty = \text{Spec } \bar{\mathbb{F}}[T^{-1}]$. We define $\tilde{\mathcal{M}}$ by specifying its sections over the open subsets $V \times U$ of $\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j$ where $V \subset \mathbb{P}_{\bar{\mathbb{F}}}^1$ and $U = \text{Spec } A \subset Y_j$ are affine open subschemes. If $V' = \text{Spec } \bar{\mathbb{F}}[T]_g \subset V_0$ for some $g \in \bar{\mathbb{F}}[T]$, then $\Gamma(V' \times U, \tilde{\mathcal{M}})$ is the pushout of $\mathfrak{M}_A \hat{\otimes}_A A[T]_g$ via the endomorphism of $M_{\bar{\mathbb{F}}} \hat{\otimes}_{\bar{\mathbb{F}}} A[T]_g$ defined by the matrix

$$C_A^0 = \begin{pmatrix} u^{(m(\mathbf{v})+x_j)/2} & T u^{(m(\mathbf{v})-x_j)/2} \\ T u^{(m(\mathbf{v})+x_j)/2} & u^{(m(\mathbf{v})-x_j)/2} \end{pmatrix}.$$

If $V'' = \text{Spec } \bar{\mathbb{F}}[T^{-1}]_h \subset V_\infty$ for some $h \in \bar{\mathbb{F}}[T^{-1}]$, then $\Gamma(V'' \times U, \widetilde{\mathcal{M}})$ is the pushout of $\mathfrak{M}_A \widehat{\otimes}_A A[T^{-1}]_h$ via the endomorphism of $M_{\bar{\mathbb{F}}} \widehat{\otimes}_{\bar{\mathbb{F}}} A[T^{-1}]_h$ defined by the matrix

$$C_A^\infty = \begin{pmatrix} T^{-1}u^{(m(\mathbf{v})+x_j)/2} & u^{(m(\mathbf{v})-x_j)/2} \\ u^{(m(\mathbf{v})+x_j)/2} & T^{-1}u^{(m(\mathbf{v})-x_j)/2} \end{pmatrix}.$$

These definitions are compatible: if $V' \subset V_0 \cap V_\infty = \text{Spec } \bar{\mathbb{F}}[T, T^{-1}]$, then the matrices C_A^0 and C_A^∞ differ by a unit (namely T resp. T^{-1}). Further this definitions are compatible with localization in the following sense.

If $U' = \text{Spec } B \subset U = \text{Spec } A$ is an affine open, then

$$\Gamma(V' \times U', \widetilde{\mathcal{M}}) = \Gamma(V' \times U, \widetilde{\mathcal{M}}) \widehat{\otimes}_A B,$$

as $\mathfrak{M}_B = \mathfrak{M}_A \widehat{\otimes}_A B$. And similarly for V'' and for localization on $\mathbb{P}_{\bar{\mathbb{F}}}^1$. As the sets $\{V' \times U, V'' \times U \mid V' \subset V_0, V'' \subset V_\infty, U \subset Y_j \text{ affine open}\}$ form a basis of the topology this indeed defines a sheaf of $\mathcal{O}_{\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j}[[u]]$ -lattices on $\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j$.

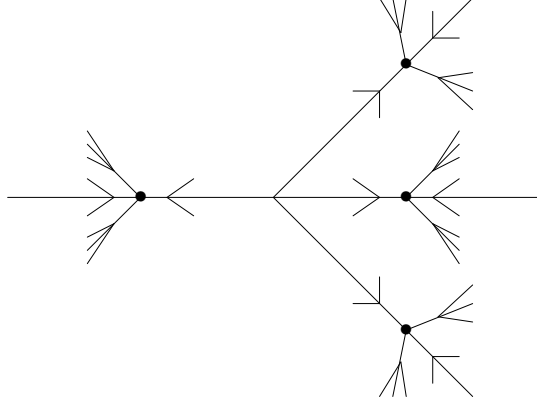


FIGURE 8. The closed points of Z_j in the building in the case $p = 3$, $\mathbb{F} = \mathbb{F}_3$. The fat points mark the points Q_j^z for $z \in \mathbb{P}^1(\mathbb{F})$.

By construction the values of \widetilde{f}_j on closed points are given by

$$\widetilde{f}_j((z_1 : z_2), x) = \psi \left(u^{(m(\mathbf{v})+x_j)/2}(z_1 e_2 + z_2 e_1), u^{(m(\mathbf{v})-x_j)/2}(z_1 e_1 + z_2 e_2) \right) (x).$$

If we set $T = z \in \bar{\mathbb{F}}$ (resp. $T^{-1} = 0$), then we pushout the Schubert variety Y_j along the automorphism

$$\begin{aligned} e_1 &\mapsto u^{(m(\mathbf{v})+x_j)/2} e_1 \\ e_2 &\mapsto u^{(m(\mathbf{v})-x_j)/2} (z e_1 + e_2). \end{aligned}$$

This is the Schubert variety consisting of the lattices \mathfrak{M} with $d_1(\mathfrak{M}, Q_j^z) \leq n_j$, where

$$Q_j^z = \langle u^{(m(\mathbf{v})+x_j)/2} e_1, u^{(m(\mathbf{v})-x_j)/2} (z e_1 + e_2) \rangle.$$

The conclusion for the point at infinity in $\mathbb{P}^1(\bar{\mathbb{F}})$ is similar. This also shows that the image of \widetilde{f}_j is Z_j . As $\mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j$ is reduced, the morphism \widetilde{f}_j factors through Z_j and we obtain a surjective morphism $f_j : \mathbb{P}_{\bar{\mathbb{F}}}^1 \times Y_j \rightarrow Z_j$. As the source of this

morphism is projective, it follows that Z_j is a closed irreducible subset of the affine Grassmannian and that the morphism f_j is projective.

We have to show that it is birational. Denote by $\tilde{U} \subset Y_j$ the subset of all lattices

$$\{\mathfrak{M} = \langle u^{n_j/2} e_1, u^{-n_j/2} (q e_1 + e_2) \mid q = \sum_{i=0}^{n_j-1} a_i u^i \rangle\},$$

(our assumptions guarantee that n_j is even in this case). This subscheme is isomorphic to the affine space $A_{\mathbb{F}}^{n_j}$ and is a maximal dimensional affine subspace of Y_j . Now $\tilde{V} = f_j(V_0 \times \tilde{U}) \subset Z_j$ is the subset of all lattices

$$\left\{ \mathfrak{M} = \langle u^{(n+2j+2)/2} e_1, u^{-(n+2j+2)/2} (q e_1 + e_2) \rangle \mid q = a_0 + \sum_{i=l+(p+1)j}^{n+2j+1} a_i u^i \right\}.$$

This is again an affine space and f_j maps $V_0 \times \tilde{U}$ isomorphically onto \tilde{V} . Thus it is birational.

The case $m(\mathbf{v}) \not\equiv x_j \pmod{2}$ is similar. We have to consider the lattices

$$\begin{aligned} [l + (p+1)j, m(\mathbf{v}) - 1]_z & \quad \text{for } z \in \bar{\mathbb{F}} \\ [-l - (p+1)j, m(\mathbf{v}) - 1]_0 \end{aligned}$$

instead of Q_j^z . Now $Y_j \hookrightarrow \text{Grass } M_{\bar{\mathbb{F}}}$ is the inclusion of the Schubert variety of lattices \mathfrak{M} with $d_2(\mathfrak{M}, \langle b_1, b_2 \rangle) = 1$ and the same condition on d_1 as above. The conclusion is now similar.

(iii) For $i \geq 0$ we always have

$$Z_i \not\subset \left(\bigcup_{j=0}^{i-1} Z_j \right) \cup Z,$$

because for example $[n + 2i + 2, m(\mathbf{v})]_0 \in Z_i(\bar{\mathbb{F}})$ but not in the latter union, as we can see from the definitions. If $l \neq 2$, then

$$Z \not\subset \bigcup_{j \geq 0} Z_j$$

because for example $[n, m(\mathbf{v})]_u \in Z(\bar{\mathbb{F}})$ but not in the latter union. The claim follows from that and the computation of the dimensions: At first consider $Z \subset X_0^Y$. This is irreducible and its complement has dimension less or equal to $\dim Z$. Now consider $Z \cup Z_0 \not\supset Z$. The complement of this subscheme has dimension (strictly) less than $\dim Z_0$. Proceeding by induction on j yields the claim.

(iv) In the case $l = 2$ we have $Z \subset Z_0$: If $\mathfrak{M} = [x, m(\mathbf{v})]_q \in Z(\bar{\mathbb{F}})$ and if we assume again $v_u(q) > 0$, then $d_1(\mathfrak{M}, T) \leq n - 1$ and hence

$$d_1(\mathfrak{M}, Q_0^0) = d_1(\mathfrak{M}, Q') + d_1(Q', Q_0^0) \leq n - 1 + l - 1 = n + 2 - l.$$

Thus each point $\mathfrak{M} \in Z(\bar{\mathbb{F}})$ is also contained in Z_0 . The statement now follows by the same argument as in (iii).

(v) This is a consequence of (i)-(iv). \square

Remark 5.3. On each of the half lines \mathcal{L}_z for $z \in \mathbb{P}^1(\bar{\mathbb{F}})$ we find Schubert varieties with decreasing dimensions. This behavior is called "*thinning tubes*" in [PR2] 6.d. compare loc. cit. B 2.

5.2. **The case** $(M_{\bar{\mathbb{F}}}, \Phi) \cong (M_1, \Phi_1) \oplus (M_2, \Phi_2)$. In this section we assume that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a, b \in \bar{\mathbb{F}}^\times$ and $0 \leq s, t < p-1$. Further we assume $s \neq t$ or $a \neq b$.

Assume $s = t$ and let n be the largest integer that is congruent to $m(\mathbf{v}) \pmod{2}$ and that satisfies

$$(5.2.1) \quad n \leq \frac{r_1 - r_2}{p+1}.$$

Denote by l the smallest integer satisfying

$$(5.2.2) \quad n + 2 \leq \frac{r_1 - r_2 + 2l}{p+1}.$$

Define the points

$$Q_j^\pm = [\pm(l + (p+1)j), m(\mathbf{v})]_0,$$

and the subschemes $Z, Z_j^\pm \subset X_0^\mathbf{v}$ by:

$$\begin{aligned} Z(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, P_{\text{red}}) \leq n\} \\ Z_j^\pm(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^\pm) \leq n + 2 - l - (p-1)j\}. \end{aligned}$$

Proposition 5.4. *Assume $s = t$ and define n and l as in (5.2.1) and (5.2.2).*

- (i) *The scheme Z is isomorphic to an n -dimensional Schubert variety.*
- (ii) *The schemes Z_j^\pm are isomorphic to $n + 2 - l - (p-1)j$ dimensional Schubert varieties.*
- (iii) *If $l \neq 1$, then*

$$X_0^\mathbf{v} = Z \cup \left(\bigcup_{j \geq 0} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0} Z_j^- \right)$$

is the decomposition of $X_0^\mathbf{v}$ into its irreducible components.

- (iv) *If $l = 1$, then*

$$X_0^\mathbf{v} = \left(\bigcup_{j \geq 0} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0} Z_j^- \right)$$

is the decomposition of $X_0^\mathbf{v}$ into its irreducible components.

- (v) *The dimension of $X_0^\mathbf{v}$ is given by*

$$\dim X_0^\mathbf{v} = \begin{cases} n+1 & \text{if } l = 1 \\ n & \text{if } l \neq 1. \end{cases}$$

Proof. (i) and (ii) follow immediately from the definitions.

As in Lemma 5.1 we easily find

$$(5.2.3) \quad X_0^\mathbf{v} = Z \cup \left(\bigcup_{j \geq 0} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0} Z_j^- \right)$$

and as in Proposition 5.2 we find

$$Z_i^\pm \not\subset Z \cup \left(\bigcup_{j \geq 0}^{i-1} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0}^{i-1} Z_j^- \right).$$

Further

$$\begin{aligned} Z &\not\subset Z_0^\pm & \text{if } l \neq 1 \\ Z &\subset Z_0^\pm & \text{if } l = 1. \end{aligned}$$

The computations are the same as in the proof of Proposition 5.2 with the only difference that we have to replace \mathcal{T} by $\mathcal{L}_0 \cup \mathcal{L}_\infty = \mathcal{A}_0 \cap \bar{\mathcal{B}}(m(\mathbf{v}))$. Part (iii) and (iv) now follow exactly as in the proof of Proposition 5.2.

Finally (v) follows from (i)-(iv). \square

In the case $s \neq t$ we have to distinguish more different cases. We only sketch the structure of the irreducible components.

Denote by $x_0 = \lfloor \frac{t-s}{p-1} \rfloor$ the integral part of $\frac{t-s}{p-1}$. Let n_+ be the largest integer congruent to $m(\mathbf{v}) \pmod 2$ such that

$$(5.2.4) \quad n_+ \leq \frac{t-s}{p-1} + \frac{1}{p+1}(r_1 - r_2 + 2(x_0 + 1 - \frac{t-s}{p-1})).$$

Let n_- be the smallest integer congruent to $m(\mathbf{v}) \pmod 2$ such that

$$(5.2.5) \quad n_- \geq \frac{t-s}{p-1} - \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - x_0)).$$

By Lemma 4.13, these numbers have the following meaning: The maximal distance d_1 of a \mathbf{v} -admissible lattice in $\mathcal{A}_q \setminus \mathcal{A}_0$ with $v_u(q) = x_0 + 1$ from P_{red} is $n_+ - \frac{t-s}{p-1}$; the maximal distance d_1 of a \mathbf{v} -admissible lattice in $\mathcal{A}_q \setminus \mathcal{A}_0$ with $v_u(q) = x_0$ from P_{red} is $\frac{t-s}{p-1} - n_-$. We define

$$(5.2.6) \quad \begin{aligned} x_1 &= \frac{1}{2}(n_+ + n_-) \\ n &= \frac{1}{2}(n_+ - n_-). \end{aligned}$$

Let Z be the subscheme whose closed points are given by

$$Z(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, [x_1, m(\mathbf{v})]_0) \leq n\}.$$

This is a n -dimensional Schubert variety. Let l_+ be the smallest integer such that

$$n_+ + 2 \leq \frac{t-s}{p-1} + \frac{1}{p+1}(r_1 - r_2 + 2(l_+ - \frac{t-s}{p-1})),$$

i.e. the smallest integer such that there are \mathbf{v} -admissible lattices with x -coordinate $n_+ + 2$ in the apartments branching of from \mathcal{A}_0 at the line $x = l_+$.

Similarly, let l_- the largest integer such that

$$n_- - 2 \geq \frac{t-s}{p-1} - \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - l_-)).$$

For $j \geq 0$ we define the following points

$$\begin{aligned} Q_j^+ &= [l_+ + (p+1)j, m(\mathbf{v})]_0 \\ Q_j^- &= [l_- - (p+1)j, m(\mathbf{v})]_0. \end{aligned}$$

Again, we define the following subschemes of X_0^\vee :

$$(5.2.7) \quad \begin{aligned} Z_j^+(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^+) \leq n_+ + 2 - l_+ - (p-1)j\} \\ Z_j^-(\bar{\mathbb{F}}) &= \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^-) \leq l_- + 2 - n_- - (p-1)j\}. \end{aligned}$$

These subschemes are isomorphic to Schubert varieties.

Lemma 5.5. *With the above notation we have*

$$(5.2.8) \quad X_0^\vee(\bar{\mathbb{F}}) = Z(\bar{\mathbb{F}}) \cup \left(\bigcup_{j \geq 0} Z_j^+(\bar{\mathbb{F}}) \right) \cup \left(\bigcup_{j \geq 0} Z_j^-(\bar{\mathbb{F}}) \right).$$

Proof. The proof of this fact is similar to the proof of Lemma 5.1.

If $\mathfrak{M} = [x, m(\mathbf{v})]_q$ is a lattice, we denote by Q' the unique point in $\mathcal{A}_0 \cap \bar{B}(m(\mathbf{v}))$ with $d_1(\mathfrak{M}, Q') = d_1(\mathfrak{M}, \mathcal{A}_0)$. We assume that $Q' = [x', m(\mathbf{v})]_0$ with $x' > \frac{t-s}{p-1}$.

If $x' < l_+$, then \mathfrak{M} is \mathbf{v} -admissible (and non- \mathbf{v} -ordinary) if and only if $\mathfrak{M} \in Z(\bar{\mathbb{F}})$. This is a direct consequence of the definitions.

If $x' \geq l_+$, there is a unique integer $j \geq 0$ such that

$$l_+ + (p+1)j \leq x' < l_+ + (p+1)(j+1).$$

In this case the definitions imply that \mathfrak{M} is \mathbf{v} -admissible (and non- \mathbf{v} -ordinary) if and only if $\mathfrak{M} \in Z_j^+(\bar{\mathbb{F}})$ (compare the proof of Lemma 5.1).

The conclusions for $x' < \frac{t-s}{p-1}$ are similar. In the set of coordinates considered above, the computations become more complicated, but we can also deduce this result by interchanging e_1 and e_2 . \square

Proposition 5.6. *With the notations of (5.2.4)-(5.2.7) :*

$$(5.2.9) \quad X_0^\mathbf{v} = Z \cup \left(\bigcup_{j \geq 0} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0} Z_j^- \right).$$

(i) *The scheme Z is isomorphic to an n -dimensional Schubert variety.*

(ii) *For $j \geq 0$ the schemes Z_j^\pm are isomorphic to Schubert varieties of dimension*

$$\begin{aligned} \dim Z_j^+ &= n_+ + 2 - l_+ - (p-1)j \\ \dim Z_j^- &= l_- + 2 - n_- - (p-1)j. \end{aligned}$$

(iii) *If $Z \not\subset Z_0^+ \cup Z_0^-$, then (5.2.9) is the decomposition of $X_0^\mathbf{v}$ into its irreducible components.*

(iv) *If $Z \subset Z_0^+ \cup Z_0^-$, then*

$$X_0^\mathbf{v} = \left(\bigcup_{j \geq 0} Z_j^+ \right) \cup \left(\bigcup_{j \geq 0} Z_j^- \right)$$

is the decomposition of $X_0^\mathbf{v}$ into its irreducible components.

In this case we would have to consider many cases in order to determine whether $Z \subset Z_0^+ \cup Z_0^-$ or not from the given integers r_1, r_2, s, t .

Proof. By use of Lemma 5.5, this is nearly the same as in the proof of Proposition 5.4. \square

Remark 5.7. Again, we find that the Schubert varieties of decreasing dimension defined above correspond to the "thinning tubes" in [PR2] 6.d along the Φ -stable half lines $\{[x, m(\mathbf{v})]_0 \mid x \leq \frac{t-s}{p-1}\}$ and $\{[x, m(\mathbf{v})]_0 \mid x \geq \frac{t-s}{p-1}\}$ in the building for $PGL_2(\bar{\mathbb{F}}((u)))$.

5.3. The case of a non split extension. As in section 4.3, we assume that there is a basis e_1, e_2 of $M_{\bar{\mathbb{F}}}$ such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} au^s & \gamma \\ 0 & bu^t \end{pmatrix}$$

for some $a, b \in \bar{\mathbb{F}}^\times$, $\gamma \in \bar{\mathbb{F}}((u))$ and $0 \leq s, t < p-1$. We assume that $(M_{\bar{\mathbb{F}}}, \Phi)$ is a non-split extension and use the notations of section 4.3.

Denote by x_0 the largest integer $x_0 < \frac{k-s}{p}$. Let n_+ be the largest integer congruent $m(\mathbf{v}) \pmod 2$ such that

$$(5.3.1) \quad n_+ \leq \frac{1}{p+1}(r_1 - r_2 - s - t + 2k).$$

Let n_- be the smallest integer congruent $m(\mathbf{v}) \pmod 2$ such that

$$(5.3.2) \quad n_- \geq \frac{t-s}{p-1} - \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - x_0)).$$

These numbers have the following meaning: The integer n_+ is the maximal x -coordinate of a \mathbf{v} -admissible lattice in \mathcal{A}_q with $v_u(q) \geq \frac{k-s}{p}$; further $\frac{t-s}{p-1} - n_-$ is the maximal distance d_1 of a \mathbf{v} -admissible lattice in $\mathcal{A}_q \setminus \mathcal{A}_0$ with $v_u(q) = x_0$ from P_{red} . As above, we define the following integers

$$(5.3.3) \quad \begin{aligned} x_1 &= \frac{1}{2}(n_+ + n_-) \\ n &= \frac{1}{2}(n_+ - n_-). \end{aligned}$$

We have $x_1 \in \{x_0, x_0 + 1\}$ which can be deduced from the equation

$$\frac{1}{p+1}(r_1 - r_2 - s - t + 2k) - \frac{k-s}{p} = \frac{k-s}{p} - (\frac{t-s}{p-1} - \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - \frac{k-s}{p}))).$$

(Here, we compute the distance from the point $\frac{k-s}{p}$ and write $\frac{k-s}{p}$ instead of x_0 as in (5.3.2)). We define the following subset

$$(5.3.4) \quad Z(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, [x_1, m(\mathbf{v})]_0) \leq n\}.$$

Let l_- be the largest integer such that

$$(5.3.5) \quad n_- - 2 \geq \frac{t-s}{p-1} - \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - l_-)).$$

For $j \geq 0$ we define the points $Q_j^- = [l_- - (p+1)j, m(\mathbf{v})]_0$ and the subsets

$$(5.3.6) \quad Z_j^-(\bar{\mathbb{F}}) = \{\mathfrak{M} \in \bar{\mathcal{B}}(m(\mathbf{v})) \mid d_1(\mathfrak{M}, Q_j^-) \leq l_- + 2 - n_- - (p-1)j\}.$$

Lemma 5.8. *With the above notations*

$$X_0^\mathbf{v}(\bar{\mathbb{F}}) = Z(\bar{\mathbb{F}}) \cup \left(\bigcup_{j \geq 0} Z_j^-(\bar{\mathbb{F}}) \right).$$

Proof. Let $\mathfrak{M} = [x, m(\mathbf{v})]_q$ be a lattice.

If $v_u(q) \geq \frac{k-s}{p}$ (or equivalently if $v_u(q) > x_0$), then (by Lemma 4.17) \mathfrak{M} is \mathbf{v} -admissible (and non- \mathbf{v} -ordinary) iff

$$x \leq \frac{1}{p+1}(r_1 - r_2 - s - t + 2k),$$

or equivalently iff $x \leq n_+$.

If $v_u(q) = x_0$ and $x > v_u(q)$, then (by Lemma 4.17) \mathfrak{M} is \mathbf{v} -admissible and non- \mathbf{v} -ordinary iff

$$d_1(\mathfrak{M}, [\frac{t-s}{p-1}, m(\mathbf{v})]_0) \leq \frac{1}{p+1}(r_1 - r_2 + 2(\frac{t-s}{p-1} - x_0)),$$

or equivalently iff $d_1(\mathfrak{M}, [\frac{t-s}{p-1}, m(\mathbf{v})]_0) \leq \frac{t-s}{p-1} - n_-$. By the definitions of x_1 and n and the fact $x_1 \in \{x_0, x_0 + 1\}$, we find that in both cases \mathfrak{M} is \mathbf{v} -admissible (and non \mathbf{v} -ordinary) iff $\mathfrak{M} \in Z(\bar{\mathbb{F}})$.

For the \mathbf{v} -admissible lattices $\mathfrak{M} = [x, m(\mathbf{v})]_q$ with $v_u(q) < x_0$ we proceed as in the proof of Lemma 5.5. \square

Proposition 5.9. *With the notations of (5.3.1)-(5.3.6):*

$$(5.3.7) \quad X_0^\vee = Z \cup \left(\bigcup_{j \geq 0} Z_j^- \right).$$

- (i) *The scheme Z is isomorphic to an n -dimensional Schubert variety.*
- (ii) *For $j \geq 0$ the schemes Z_j^- are isomorphic to Schubert varieties of dimension*

$$\dim Z_j^- = l_- + 2 - n_- - (p-1)j.$$

- (iii) *If $Z \not\subset Z_0^-$, then (5.3.7) is the decomposition of X_0^\vee into its irreducible components.*
- (iv) *If $Z \subset Z_0^-$, then the decomposition of X_0^\vee into its irreducible components is given by*

$$X_0^\vee = \bigcup_{j \geq 0} Z_j^-.$$

Proof. (i),(ii) This follows from the definitions.

(iii),(iv) As in the discussion of the other cases, our Schubert varieties are constructed in a way such that

$$Z_i^- \not\subset Z \cup \left(\bigcup_{j=0}^{i-1} Z_j^- \right)$$

for all $i \geq 0$. The claim follows from this and the computation of the dimension (compare the proof of Proposition 5.2). \square

Remark 5.10. We find a sequence of Schubert varieties of decreasing dimension along the unique Φ -stable half line $\{[x, m(\mathbf{v})]_0 \mid x < \frac{k-s}{p}\}$ corresponding to the "thinning tubes" in [PR2] 6.d.

Further, we find a Schubert variety Z corresponding to a ball with given radius around a given point as in loc. cit. A 3 resp. B 2.

The discussion of this section implies the following result.

Theorem 5.11. *If $(M_{\mathbb{F}}, \Phi)$ is not isomorphic to the direct sum of two isomorphic one-dimensional ϕ -modules, then the irreducible components of X_0^\vee are Schubert varieties. Especially they are normal.*

6. RELATION TO RAYNAUD'S THEOREM

In this section, we assume $p \neq 2$. In [Ra], Raynaud introduces a partial order on the set of finite flat models for $V_{\mathbb{F}}$ (i.e. the set of \mathbb{F} -valued points of $\mathcal{GR}_{V_{\mathbb{F}},0}$) by defining $\mathcal{G}_1 \preceq \mathcal{G}_2$ if there exists a morphism $\mathcal{G}_2 \rightarrow \mathcal{G}_1$ inducing the identity on the generic fiber of $\mathrm{Spec} \mathcal{O}_K$. By (loc. cit. 2.2.3 and 3.3.2), this order admits a minimal and maximal object (if the set is non-empty) which agree if $e < p-1$.

In our case, Raynaud's partial order is given by the inclusion of lattices in $M_{\mathbb{F}}$: Inclusion of two admissible lattices is a morphism that commutes with the semi-linear map Φ and induces the identity of $M_{\mathbb{F}}$ after inverting u . Here, a lattice \mathfrak{M} is called *admissible* if it defines a finite flat group scheme, i.e. if $u^e \mathfrak{M} \subset \langle \Phi(\mathfrak{M}) \rangle \subset \mathfrak{M}$.

Proposition 6.1. *Let $\rho : G_K \rightarrow V_{\mathbb{F}}$ be a continuous representation of G_K . Assume that there exists a finite extension \mathbb{F}' of \mathbb{F} such that there is a finite flat group scheme*

model over $\text{Spec } \mathcal{O}_K$ for the induced G_K representation on $V_{\mathbb{F}'} = V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. Then there exists a finite flat model for $V_{\mathbb{F}}$, i.e.

$$\mathcal{GR}_{V_{\mathbb{F}},0} \neq \emptyset \Rightarrow \mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}) \neq \emptyset.$$

Proof. Our assumptions imply $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}') \neq \emptyset$ for some finite extension \mathbb{F}' of \mathbb{F} . Hence, by Raynaud's theorem, the set $\mathcal{GR}_{V_{\mathbb{F}},0}(\mathbb{F}')$ has a unique minimal element. The natural action of the Galois group $\text{Gal}(\mathbb{F}'/\mathbb{F})$ on $\mathcal{GR}_{V_{\mathbb{F}},0}^{\text{loc}}(\mathbb{F}')$ preserves the partial order (it preserves inclusion of lattices) and hence the minimal element is stable under this action. Consequently, the minimal object is already defined over \mathbb{F} . \square

We now want to reprove Raynaud's theorem in our context: we will show that there is a minimal and a maximal lattice for the order induced by inclusion on the set $\mathcal{GR}_{V_{\mathbb{F}},0}(\overline{\mathbb{F}})$.

Proposition 6.2. *There exists a minimal and a maximal admissible lattice \mathfrak{M}_{\min} resp. \mathfrak{M}_{\max} for the order defined by the inclusion.*

Proof. We only prove the statement about the maximal lattice. The other one is analogue. We choose a basis e_1, e_2 . The proposition follows from the following two observations:

- (a) *There exists a unique admissible lattice with minimal y -coordinate.*
- (b) *If \mathfrak{M} is a admissible lattice with non-minimal y -coordinate, then it is contained in an admissible lattice with strictly smaller y -coordinate.*

Proof of (a): First it is clear that the y -coordinates of admissible lattices are bounded below: If

$$\langle e_1, e_2 \rangle \sim A' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

and if $\mathfrak{M} = [x, y]_q$ is admissible, then $2e - d' = (p-1)y + v_u(\det A')$ with

$$0 \leq d' = \dim \langle \Phi(\mathfrak{M}) \rangle / u^e \mathfrak{M} \leq 2e.$$

Suppose now that \mathfrak{M}_1 and \mathfrak{M}_2 are admissible lattices with the same y -coordinate. There is a basis b_1, b_2 such that

$$\begin{aligned} \mathfrak{M}_1 &= \langle b_1, b_2 \rangle \\ \mathfrak{M}_2 &= \langle u^n b_1, u^{-n} b_2 \rangle. \end{aligned}$$

for some $n \geq 0$. We have

$$\begin{aligned} \mathfrak{M}_1 &\sim A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \mathfrak{M}_2 &\sim B = \begin{pmatrix} u^{n(p-1)}\alpha & u^{-n(p+1)}\beta \\ u^{n(p+1)}\gamma & u^{-n(p-1)}\delta \end{pmatrix} \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{F}}((u))$. Define

$$\mathfrak{M}_3 = \langle b_1, u^{-n} b_2 \rangle = \begin{pmatrix} 1 & 0 \\ 0 & u^{-n} \end{pmatrix} \mathfrak{M}_1.$$

Then \mathfrak{M}_3 has strictly smaller y -coordinate and is admissible. Indeed:

$$\mathfrak{M}_3 \sim C = \begin{pmatrix} \alpha & u^{-np}\beta \\ u^n\gamma & u^{-n(p-1)}\delta \end{pmatrix}$$

and we have to show: $\min_{i,j} c_{ij} \geq 0$ and $v_u(\det C) - \min_{i,j} c_{ij} \leq e$.
But because \mathfrak{M}_1 and \mathfrak{M}_2 are admissible we know:

$$\begin{aligned} v_u(\alpha) &\geq 0 & v_u(u^{-np}\beta) &\geq n \\ v_u(u^n\gamma) &\geq n & v_u(u^{-n(p-1)}\delta) &\geq 0. \end{aligned}$$

Similarly $v_u(\det C) = v_u(\det A) - (p-1)n = v_u(\det B) - (p-1)n$ and hence:

$$\begin{aligned} v_u(\det C) - v_u(\alpha) &\leq e - (p-1)n & v_u(\det C) - v_u(u^{-np}\beta) &\leq e - pn \\ v_u(\det C) - v_u(u^n\gamma) &\leq e - pn & v_u(\det C) - v_u(u^{-n(p-1)}\delta) &\leq e - (p-1)n. \end{aligned}$$

Proof of (b): Let \mathfrak{M} be an admissible lattice with non-minimal y -coordinate. Then there exists an admissible lattice \mathfrak{M}' with strictly smaller y -coordinate. These lattices are contained in a common apartment and hence there is a basis b_1, b_2 such that

$$\begin{aligned} \mathfrak{M} &= \langle b_1, b_2 \rangle \\ \mathfrak{M}' &= \langle u^m b_1, u^n b_2 \rangle \end{aligned}$$

for some integers m, n with $m + n < 0$, because the y -coordinate of \mathfrak{M}' is strictly smaller than the y -coordinate of \mathfrak{M} .

Without loss of generality, we assume $m - n \geq 0$. If $m \leq 0$, then $n \leq 0$ and we are done, since then $\mathfrak{M} \subset \mathfrak{M}'$.

If $m > 0$ we claim that the lattice $\mathfrak{M}_1 = \langle b_1, u^{m+n}b_2 \rangle$ is admissible. Indeed

$$\begin{aligned} \mathfrak{M} &\sim \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ \mathfrak{M}' &\sim \begin{pmatrix} u^{(p-1)m}\alpha & u^{pn+m}\beta \\ u^{pm-n}\gamma & u^{(p-1)n}\delta \end{pmatrix} \\ \mathfrak{M}_1 &\sim \begin{pmatrix} \alpha & u^{p(m+n)}\beta \\ u^{-m-n}\gamma & u^{(p-1)(m+n)}\delta \end{pmatrix} \end{aligned}$$

and the claim follows by a similar argument as in the proof of (a). \square

Proposition 6.3. *If $e < p-1$, then the minimal and the maximal lattice coincide.*

Proof. Denote the minimal lattice by \mathfrak{M}_{\min} , the maximal by \mathfrak{M}_{\max} . There is a apartment containing both lattices and we may assume $\mathfrak{M}_{\max} = \langle e_1, e_2 \rangle = [0, 0]_0$ and $\mathfrak{M}_{\min} = [x, y]_0$ for some $y \geq 0$.

Let $A \in GL_2(\mathbb{F}((u)))$ be a matrix such that

$$\mathfrak{M}_{\max} \sim A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Define $d_{\min} = \dim \mathfrak{M}_{\min} / \langle \Phi(\mathfrak{M}_{\min}) \rangle$ and similarly d_{\max} . Then

$$\begin{aligned} 2e - d_{\max} &= 2e - \dim \mathfrak{M}_{\max} / \langle \Phi(\mathfrak{M}_{\max}) \rangle = v_u(\det A) \\ 2e - d_{\min} &= 2e - \dim \mathfrak{M}_{\min} / \langle \Phi(\mathfrak{M}_{\min}) \rangle = v_u(\det A) + (p-1)y. \end{aligned}$$

Thus we have $(p-1)y = d_{\min} - d_{\max} \leq 2e < 2(p-1)$ and hence $y = 0$ or $y = 1$. If $y = 0$ we are done, as \mathfrak{M}_{\max} is the unique lattice with minimal y -coordinate.

Assume $y = 1$. In this case $\mathfrak{M}_{\min} \subset \mathfrak{M}_{\max}$ implies $\mathfrak{M}_{\min} = [\pm 1, 1]_0$. Without loss of generality, we assume $\mathfrak{M}_{\min} = [-1, 1]_0 = \langle ue_1, e_2 \rangle$. Then

$$\mathfrak{M}_{\min} \sim B = \begin{pmatrix} u^{p-1}\alpha & u^{-1}\beta \\ u^p\gamma & \delta \end{pmatrix}.$$

As both lattices are admissible, we have

$$\max\{v_u(\det B) - v_u(u^{-1}\beta), v_u(\det B) - v_u(\delta)\} \leq e$$

and hence:

$$\begin{aligned} v_u(\alpha) &\geq 0 & v_u(\beta) - 1 &\geq v_u(\det B) - e = v_u(\det A) + (p-1) - e > v_u(\det A) \\ v_u(\gamma) &\geq 0 & v_u(\delta) &\geq v_u(\det B) - e = v_u(\det A) + (p-1) - e > v_u(\det A) \end{aligned}$$

It follows that

$$v_u(\det A) = v_u(\alpha\delta - \beta\gamma) \geq \min\{v_u(\alpha) + v_u(\delta), v_u(\beta) + v_u(\gamma)\} > v_u(\det A).$$

Contradiction. \square

Finally we want to determine the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} for the minimal and the maximal lattice in the cases where $(M_{\bar{\mathbb{F}}}, \Phi)$ is simple resp. split reducible. If $(M_{\bar{\mathbb{F}}}, \Phi)$ is non-split reducible, the computations turn out to be very difficult and are omitted.

6.1. The absolutely simple case. As in section 3, we fix a basis e_1, e_2 such that

$$M_{\bar{\mathbb{F}}} \sim \begin{pmatrix} 0 & au^s \\ 1 & 0 \end{pmatrix}$$

with $a \in \bar{\mathbb{F}}^\times$ and $0 \leq s < p^2 - 1$. Let \mathfrak{M}_{\min} be the minimal and \mathfrak{M}_{\max} be the maximal lattice. Denote by s_1, s_2 the unique integers $0 \leq s_1 < p+1$ resp. $0 \leq s_2 < p-1$ with

$$\begin{aligned} s_1 &\equiv s \pmod{p+1} \\ s_2 &\equiv s \pmod{p-1}. \end{aligned}$$

Because $p-1$ and $p+1$ are both even, we find $s_1 \equiv s_2 \pmod{2}$.

Similarly let s'_2 be the unique integer $0 \leq s'_2 < p-1$ with $2e-s \equiv s'_2 \pmod{p-1}$.

Proposition 6.4. Denote by $m = \frac{s-s_1}{p+1}$ the integral part of $\frac{s}{p+1}$ and by $l = \frac{s-s_2}{p-1}$ resp. $l' = \frac{2e-s-s'_2}{p-1}$ the integral part of $\frac{s}{p-1}$ resp. $\frac{2e-s}{p-1}$.

(i) The elementary divisors (a_{\max}, b_{\max}) of $\langle \Phi(\mathfrak{M}_{\max}) \rangle$ with respect to \mathfrak{M}_{\max} are given by

$$\begin{cases} \left(\frac{s_1+s_2}{2}, \frac{s_2-s_1}{2} \right) & \text{if } l+m \in 2\mathbb{Z} \text{ and } s_2 \geq s_1 \\ \left(\frac{s_2-s_1}{2} + p, \frac{s_1+s_2}{2} - 1 \right) & \text{if } l+m \in 2\mathbb{Z} \text{ and } s_2 < s_1 \\ \left(\frac{s_2-s_1+(p+1)}{2}, \frac{s_1+s_2-(p-1)}{2} \right) & \text{if } l+m \notin 2\mathbb{Z} \text{ and } s_1+s_2 \geq p+1 \\ \left(\frac{s_1+s_2+(p-1)}{2}, \frac{s_2-s_1+(p-1)}{2} \right) & \text{if } l+m \notin 2\mathbb{Z} \text{ and } s_1+s_2 < p+1. \end{cases}$$

(ii) The elementary divisors (a_{\min}, b_{\min}) of $\langle \Phi(\mathfrak{M}_{\min}) \rangle$ with respect to \mathfrak{M}_{\min} are given by

$$\begin{cases} (e + \frac{s_1 - s'_2}{2}, e - \frac{s_1 + s_2}{2}) & \text{if } l' + m \in 2\mathbb{Z} \text{ and } s_1 \leq s'_2 \\ (e + 1 - \frac{s_1 + s_2}{2}, e - p + \frac{s_1 - s_2}{2}) & \text{if } l' + m \in 2\mathbb{Z} \text{ and } s_1 > s'_2 \\ (e + \frac{(p+1) - s_1 - s'_2}{2}, e + \frac{s_1 - s'_2 - (p+1)}{2}) & \text{if } l' + m \notin 2\mathbb{Z} \text{ and } s_1 + s'_2 \geq p + 1 \\ (e + \frac{s_1 - s'_2 - (p-1)}{2}, e + \frac{-s_1 - s'_2 - (p-1)}{2}) & \text{if } l' + m \notin 2\mathbb{Z} \text{ and } s_1 + s_2 < p + 1. \end{cases}$$

Proof. We only prove the statement on \mathfrak{M}_{\max} . From Corollary 3.10 we know that the lattice \mathfrak{M}_{\max} is contained in the apartment defined by e_1, e_2 . For a lattice \mathfrak{M} , denote the elementary divisors of $\langle \Phi(\mathfrak{M}) \rangle$ with respect to \mathfrak{M} by (a, b) . Then \mathfrak{M} is admissible if $0 \leq b, a \leq e$. As we are assuming that there exist admissible lattices we only have to check the condition $a, b \geq 0$ for \mathfrak{M}_{\max} (and the condition $a, b \leq e$ for \mathfrak{M}_{\min}).

If $l + m \in 2\mathbb{Z}$, the candidate for the maximal lattice is $[m, -l]_0$. If this is not admissible, then we take $[m + 1, -l + 1]_0$. Computing the elementary divisors a, b by use of Lemma 3.2 and Definition 2.2 we find the above expressions. In the case $l + m \notin 2\mathbb{Z}$ we deal with the lattices $[m + 1, -l]_0$ and $[m, -l + 1]_0$. \square

6.2. The split reducible case. As in section 4, we fix a basis e_1, e_2 such that

$$M_{\mathbb{F}} \sim \begin{pmatrix} au^s & 0 \\ 0 & bu^t \end{pmatrix}$$

with $a, b \in \mathbb{F}^\times$ and $0 \leq s, t < p - 1$.

Proposition 6.5. (i) The elementary divisors (a_{\max}, b_{\max}) of $\langle \Phi(\mathfrak{M}_{\max}) \rangle$ with respect to \mathfrak{M}_{\max} are given by

$$(a_{\max}, b_{\max}) = \begin{cases} (t, s) & \text{if } t \geq s \\ (s, t) & \text{if } s \geq t. \end{cases}$$

(ii) The elementary divisors (a_{\min}, b_{\min}) of $\langle \Phi(\mathfrak{M}_{\min}) \rangle$ with respect to \mathfrak{M}_{\min} are given by

$$(a_{\min}, b_{\min}) = \begin{cases} ((p-1)\lfloor \frac{e-t}{p-1} \rfloor + t, (p-1)\lfloor \frac{e-s}{p-1} \rfloor + s) & \text{if } \frac{t-s}{p-1} \geq \lfloor \frac{e-s}{p-1} \rfloor - \lfloor \frac{e-t}{p-1} \rfloor \\ ((p-1)\lfloor \frac{e-s}{p-1} \rfloor + s, (p-1)\lfloor \frac{e-t}{p-1} \rfloor + t) & \text{if } \frac{t-s}{p-1} \leq \lfloor \frac{e-s}{p-1} \rfloor - \lfloor \frac{e-t}{p-1} \rfloor. \end{cases}$$

Proof. From Corollary 4.12 and Corollary 4.16 we know that the minimal and the maximal lattice are contained in the apartment defined by e_1, e_2 . Now

$$\begin{aligned} \Phi(u^m e_1) &= au^{(p-1)m+s}(u^m e_1) \\ \Phi(u^n e_2) &= bu^{(p-1)n+t}(u^n e_2). \end{aligned}$$

The first part of the Proposition follows from the fact that s (resp. t) are the smallest positive integers that are congruent to s (resp. t) modulo $p - 1$.

The second part follows from the fact that $(p-1)\lfloor \frac{e-s}{p-1} \rfloor + s$ is the largest integer smaller than e that is congruent to s modulo $p - 1$ (and similar for t). \square

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